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NUMERICAL SOLUTIONS OF TURBULENT
BUOYANT JET PROBLEMS

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Frontispiece. A Buoyant Jet in a Laboratory Tank, Illustrating How the Ambient Density Stratification Prevents the Jet from Reaching the Surface. (For detailed data for this experiment, see Jet No. 16, pp. 61 and 69, in Fan (1).)

NUMERICAL SOLUTIONS OF TURBULENT BUOYANT JET PROBLEMS

by

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U. S. Public Health Service

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ABSTRACT

Theoretical solutions were obtained on four classes of turbulent buoyant jet problems, namely,

- 1) an inclined, round buoyant jet in a stagnant, uniform ambient fluid;
- 2) an inclined, round buoyant jet in a stagnant ambient fluid with linear density-stratification;
- 3) an inclined, slot buoyant jet in a stagnant, uniform ambient fluid;
- 4) an inclined, slot buoyant jet in a stagnant ambient fluid with linear density-stratification.

This report is a summary of the numerical solutions on buoyant jets in stagnant environments carried out in connection with previous investigations by Fan (10), Fan and Brooks (12) and Brooks and Koh (8). Using the integral type of analysis, assuming similarity, predictions can be made for jet trajectory, widths, and dilution ratios, in a uniform or density-stratified environment without ambient currents. Numerical solutions have been presented in dimensionless form for a wide range of initial conditions including the effect of the initial angle of discharge.

Problems with non-linear density profiles are not readily treated in generalized non-dimensional form. Rather it is more feasible to make case by case calculations using dimensional variables. A program for such calculations for a round jet is available in a technical memorandum by Ditmars (16).

These solutions are useful in the design of disposal systems for sewage effluent into the ocean or cooling water into a lake.

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CHAPTER I

INTRODUCTION

Disposal of sewage into the ocean and lakes has been practiced by many coastal cities around the world. Inadequate dispersal of the pollutants has often resulted in serious contamination of the coastal areas. Controlling of such pollution problems relies on a clear understanding of the basic flow phenomena involved.

The recent practice of marine disposal is to discharge the sewage effluent in jets through numerous ports widely spaced along the ocean outfall diffusers deep under the surface into the environment as indicated by Rawn, Bowerman and Brooks (27) and Brooks (7). The initial mixing of the sewage effluent with the sea water is induced by the turbulent jet motion. The basic flow phenomenon can be regarded as a submerged turbulent jet.

The simplest case of such turbulent jet flows is an ordinary momentum jet (or a simple jet) where the fluids involved are of identical density. But in marine disposal problems, the following environmental conditions must be taken into consideration:

1. Buoyancy effect

The density of the sewage effluent is different from the density of the sea water or brackish estuary water. Since sewage effluent has nearly the same density as fresh water, it is about 2.6% lighter than sea water. Although the difference is small, the buoyancy effect on the jet behavior is drastic. For example, a jet discharging horizontally (or obliquely) into a heavier fluid will be deflected upwards. Such a jet, which has initial buoyancy flux as well as momentum flux, is called a buoyant jet (or forced plume). The limiting case where the flow is generated solely by buoyancy flux is called a simple plume.

Buoyant jets are also produced in fresh water lakes by discharge of warm water used for cooling in industrial processes and steam generation of electric power.

2. Density stratification

Oceans and lakes are frequently stratified in density due to non-uniform temperature and/or salinity. In a stably stratified ocean, it is possible to prevent rising sewage jets from reaching the surface by inducing rapid mixing of the buoyant jets with heavy bottom water, thus producing a neutrally buoyant cloud (Brooks (7)). The submergence of the sewage field is often the most favorable situation for pollution control in the open ocean. To be able to predict the conditions for the submergence of the jet flow field and the dilution in the cloud produced is of great importance in problems of disposal of waste water.

3. Current effect

The ocean, like the atmosphere, is seldom stagnant. The ocean currents and regular tidal motions affect not only the movement of the sewage field established at or near the surface, but also the initial jet mixing characteristics. The waste gas disposal through smoke stacks is a similar problem. The effect can not be overlooked even if the current velocity is only a fraction of the jet discharge velocity (Fan (10)).

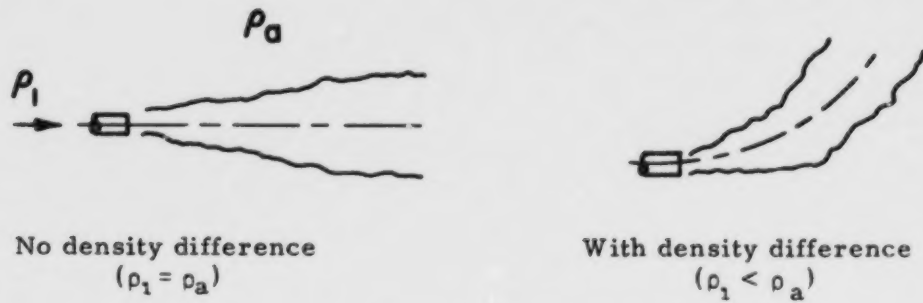
Schematic diagrams demonstrating these environmental effects are shown in Fig. 1. These three effects are considered to be the most important in considering the jet behavior. Two other factors are described below.

4. Ambient turbulence

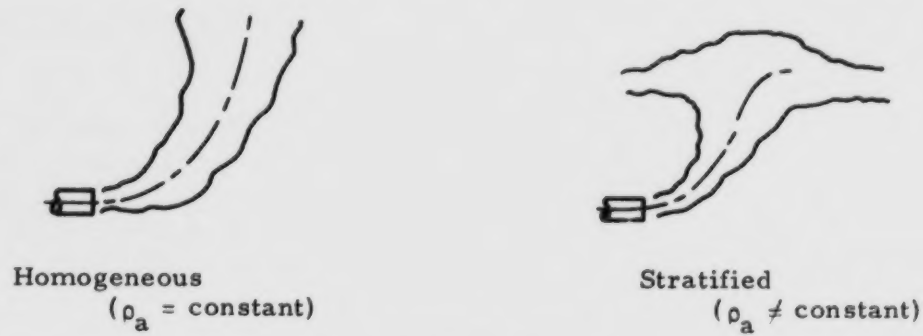
Turbulence in the environment also affects the jet behavior. However, the effects on initial jet mixing are believed to be of secondary importance unless the jet motion is relatively weak. There is little information on this subject.

5. Jet interference

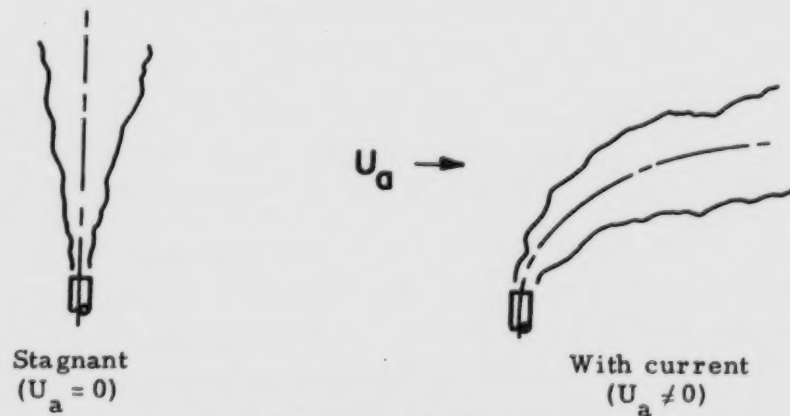
When jets are closely spaced in a row they will gradually merge as they spread. For example, a row of round jets behaves like a jet from a two-dimensional slot far away from the source. Interference effects may be anticipated by prediction of the rate of growth of the diameter of a jet.



Buoyancy Effect



Stratification Effect



Current Effect

Fig. 1. Effects of the Environmental Conditions on the Jet Behavior.

It is difficult to solve a general problem including all these factors. Buoyant jet (or plume) problems have been under extensive study by numerous investigators from various fields, as listed in the Reference List of this report. (For a detailed discussion, see Fan (10), Chapters II and IX). In this report, numerical solutions were carried out for four specific classes of buoyant jet problems, namely,

- 1) an inclined round buoyant jet in a stagnant, uniform ambient fluid;
- 2) an inclined round buoyant jet in a stagnant ambient fluid with linear density stratification;
- 3) an inclined two-dimensional slot buoyant jet in a stagnant, uniform ambient fluid;
- 4) an inclined two-dimensional slot buoyant jet in a stagnant ambient fluid with linear density stratification.

These solutions on jet gross behavior will be useful not only to marine disposal problems but also to the problems of a similar nature such as the disposal of hot water from a thermal power plant into a lake or artificial pumped mixing of a density-stratified reservoir.

Chapter II gives a summary of the basic assumptions underlying the analyses and flow configurations for all the problems.

Chapters III through VI are theoretical solutions of these four classes of problems over a wide range of initial conditions pertinent to practical applications. A summary is given in Chapter VII.

For solving problems with non-linear stratification, the dimensionless-variable approach is not convenient. It is recommended that calculations be made with dimensional variables by means of the computer program developed by Ditmars (16) on a case-by-case basis. His program is good for any arbitrary ambient density profile, and both buoyant and sinking round jets.

CHAPTER II

FLOW CONFIGURATIONS AND BASIC ASSUMPTIONS

In this chapter, the assumptions involved in the analyses are summarized. The general assumptions common to all four classes of problems are presented first. The next two sections give the flow configurations and specific assumptions for round and two-dimensional slot jets respectively.

II-A. General Assumptions

The general assumptions underlying the analyses made in this investigation are listed as follows:

1. The fluids are incompressible.
2. Variations of fluid density throughout the flow field are small compared with the reference density chosen. The variation of density can be neglected in considering inertia terms but it must be included in gravity terms. Since the variation in density is assumed small, this leads to the approximation that the conservation of mass flux can be replaced by the conservation of volume flux. This is commonly called Boussinesq assumption.
3. Within the range of variation, the density of the fluid is assumed to be a linear function of either salt concentration or heat content above the reference level.
4. Flow is fully turbulent. Molecular transport can be neglected in comparison with turbulent transport. There is no Reynolds number dependence.
5. Longitudinal turbulent transport is small compared with longitudinal convective transport.
6. Pressure is hydrostatic throughout the flow field.

7. Curvature of the trajectory of the jet is small. In other words, the ratio of the local characteristic width of the jet to the radius of curvature is small. The effect of curvature will be neglected.

8. The velocity profiles are similar at all cross sections normal to the jet trajectory. Similarity is also presumed for profiles of buoyancy and concentration of any tracer. The specific forms of the profiles are given in the next two sections. Thus, the analyses apply only to the zone of established flow where all the profiles are fully developed. However, for practical applications, the initial conditions must be adjusted to take account of the zone of flow establishment. The results by Albertson, Dai, Jensen and Rouse (5) for this region will be adopted in application to practical problems as indicated in the following chapters.

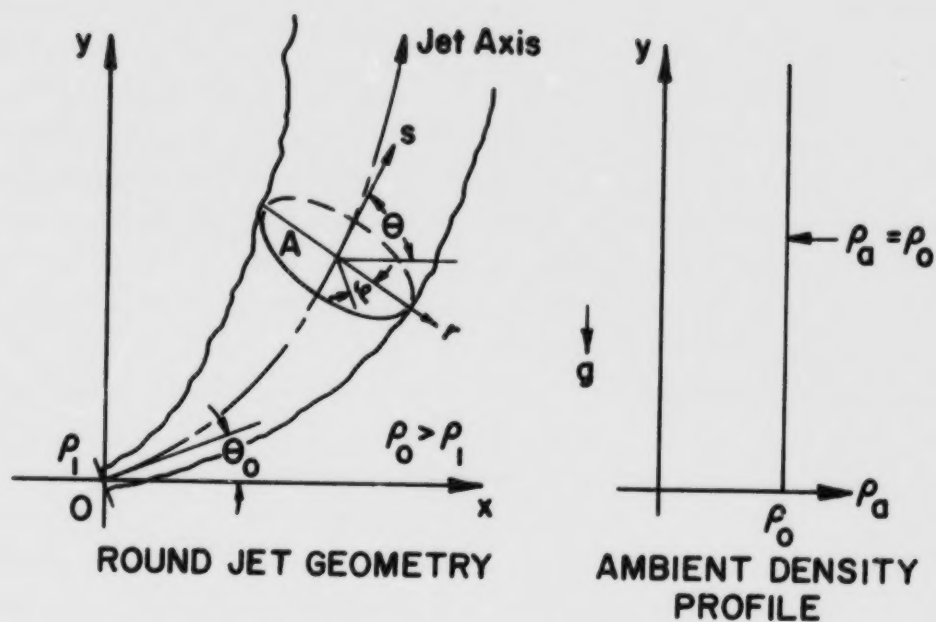
II-B. An Inclined Round Buoyant Jet in a Stagnant Environment-Uniform or with Linear Density-Stratification

Fig. 2a shows a round buoyant jet in a stagnant uniform environment and Fig. 2b in a stagnant environment with linear density stratification.

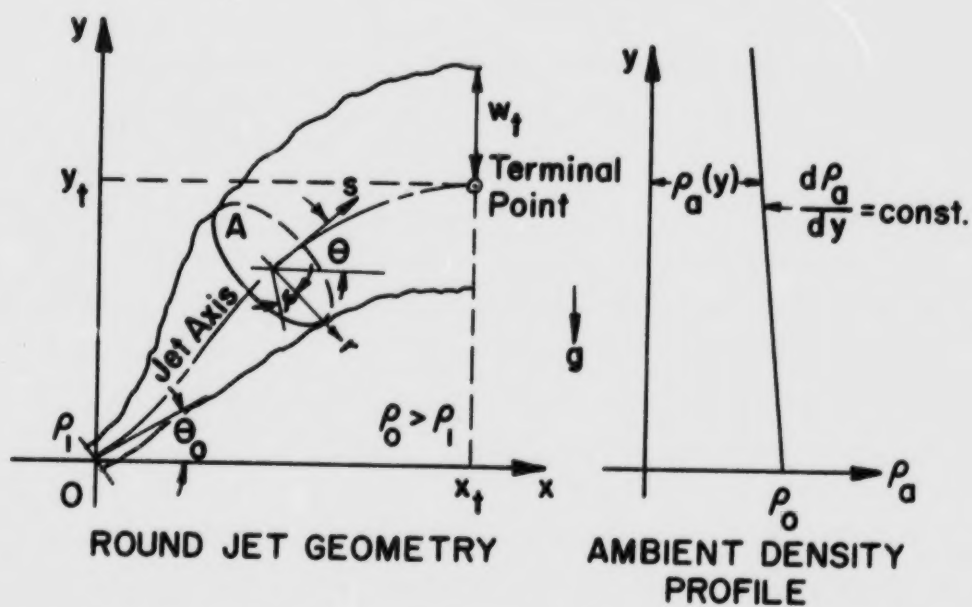
In both cases the jets are issuing from the origin at an angle of inclination θ_0 with the horizontal. The axis of the jet is taken as a parametric coordinate axis s . The angle between the s -axis and the horizontal is denoted as θ . The radial distance to the s -axis at a normal cross section A is chosen to be the r -coordinate. The angular coordinate ϕ is denoted as shown in the figures.

u^* and ρ^* are respectively local mean velocity and density which are in general functions of r , s , and ϕ , while u and p are characteristic velocity and density at the s -axis and are functions only of s . The corresponding ambient density values are similarly denoted as ρ_a^* and ρ_a . In a uniform ambient fluid $\rho_a^* = \rho_a = \text{constant}$.

In a uniform environment, the jet axis is deflected upwards because of the increase of vertical momentum flux due to the action of buoyancy force. The jet grows as it rises and entrains ambient fluid.



a) In a Uniform Environment



b) In a Linearly Density-Stratified Environment

Fig. 2. Schematic Diagram of Round Buoyant Jet Problems Studied.

In a linearly density-stratified environment, the jet axis is first deflected upwards again because of the increase of vertical momentum flux due to the action of the buoyancy force. Because of the turbulent mixing the jet entrains the denser ambient fluid and grows heavier with reduction of the driving buoyancy force. Since the density of the ambient fluid decreases with height, the jet will eventually become as heavy as, and then heavier than, the ambient fluid at the same height. The buoyancy force thus reverses its direction and in the end will stop the rising of the jet at a terminal point (x_t, y_t) where the vertical momentum flux vanishes. The trajectory of the jet is therefore in general an S-shaped curve. After reaching the terminal point the horizontal momentum flux will keep the jet moving in the x-direction. But the flow cannot maintain the characteristics of a turbulent jet soon after reaching the terminal level and collapses in the vertical direction because of the suppression of vertical motion imposed by the density stratification. The analysis will not cover that part.

Specific assumptions related to the analyses of round buoyant jet problems are listed as follows:

1. The entrainment relation is given by the equation:

$$\frac{dQ}{ds} = 2\pi\alpha b u \quad (1)$$

where Q is the volume flux across the jet cross section A ; α is a constant coefficient of entrainment for a round buoyant jet; b is the characteristic length defined in eq. (2); u is the characteristic velocity along s -axis.

2. Velocity profiles are assumed to be Gaussian, with no dependence on φ -coordinate:

$$u^*(s, r, \varphi) = u(s) e^{-r^2/b^2} \quad (2)$$

where $b = b(s)$ is a characteristic length defined by the velocity profile. Commonly $w = \sqrt{2} b$ is defined to be the nominal half width of the jet.

3. Profiles of density deficiency with respect to the ambient density are assumed to be Gaussian, with no dependence on φ .

$$\frac{\rho_o - \rho^*(s, r, \varphi)}{\rho_o} = \frac{\rho_o - \rho(s)}{\rho_o} e^{-r^2 / (\lambda b)^2}$$

(in a uniform environment) (3a)

$$\frac{\rho_a^*(s, r, \varphi) - \rho^*(s, r, \varphi)}{\rho_o} = \frac{\rho_a(s) - \rho(s)}{\rho_o} e^{-r^2 / (\lambda b)^2}$$

(in a linearly density-stratified environment) (3b)

where λb is the characteristic length of the profiles; λ^2 is the turbulent Schmidt number which is assumed to be constant and is usually found to be somewhat larger than 1. Such profiles can also be regarded as buoyancy profiles.

4. Profiles of a certain tracer concentration are also similar and assumed to be Gaussian:

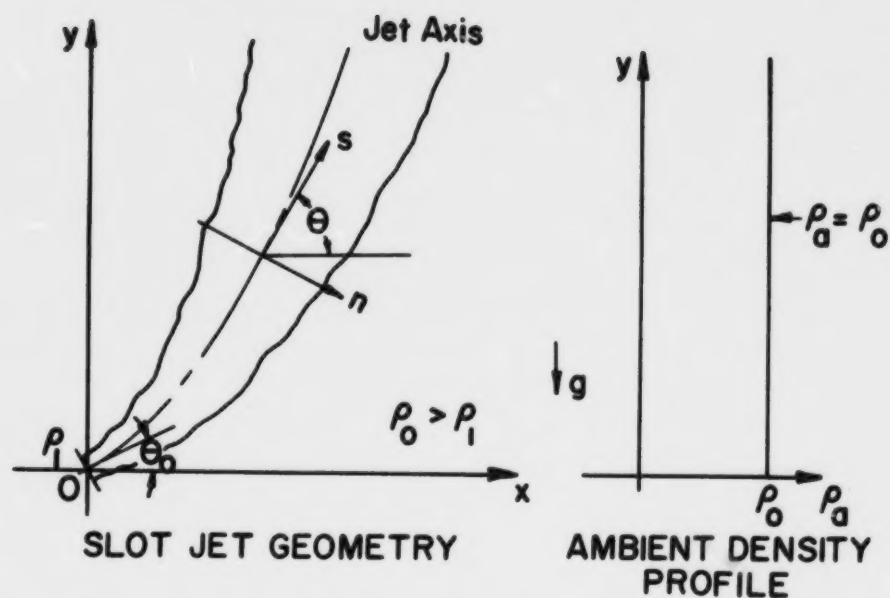
$$c^*(s, r, \varphi) = c(s) e^{-r^2 / (\lambda b)^2} \quad (4)$$

II-C. An Inclined Two-Dimensional Slot Buoyant Jet in a Stagnant Environment — Uniform or with Linear Density Stratification

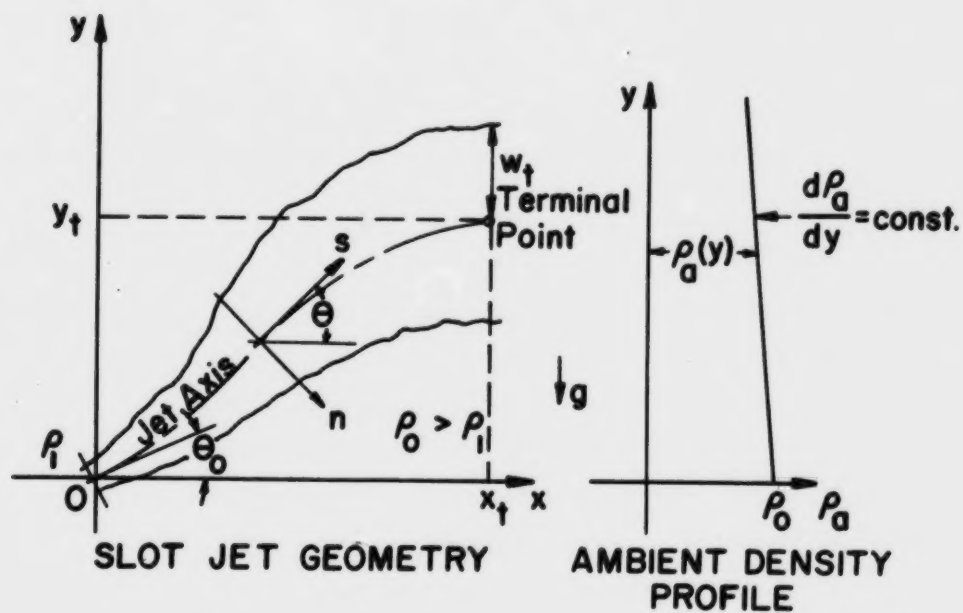
Fig. 3a shows a two-dimensional slot jet in a stagnant, uniform environment and Fig. 3b in a stagnant environment with linear density stratification.

In both cases, the jets are issuing from the z-axis at an angle of inclination θ_o with the horizontal. The axis of the jet is again taken as a parametric coordinate axis s . The distance normal to the s -axis is taken to be n -coordinate as shown.

The flow configurations are entirely similar to those described in Section II-B. These will not be repeated here. Specific assumptions related to the analyses of two-dimensional slot jet problems are listed as follows:



a) In a Uniform Environment



b) In a Linearly Density-Stratified Environment

Fig. 3. Schematic Diagram of Slot Buoyant Jet Problems Studied.

1. The entrainment relation is given by the equation:

$$\frac{dq}{ds} = 2\alpha u \quad (5)$$

where q is the volume flux per unit length along z -axis, α is a constant entrainment coefficient for a slot buoyant jet.

2. Velocity profiles are assumed to be Gaussian:

$$u^*(s, n) = u(s) e^{-n^2/b^2} \quad (6)$$

$\sqrt{2} b$ is again defined to be the nominal half width of the jet.

3. Profiles of density deficiency with respect to the ambient density are assumed to be Gaussian:

$$\frac{\rho_o - \rho^*(s, n)}{\rho_o} = \frac{\rho_o - \rho(s)}{\rho_o} e^{-n^2/(\lambda b)^2} \quad (7a)$$

(in a uniform environment)

$$\frac{\rho_a^*(s, n) - \rho^*(s, n)}{\rho_o} = \frac{\rho_a(s) - \rho(s)}{\rho_o} e^{-n^2/(\lambda b)^2}$$

(in a linearly density-stratified environment) (7b)

4. Profiles of a certain tracer concentration are also Gaussian:

$$c^*(s, n) = c(s) e^{-n^2/(\lambda b)^2} \quad (8)$$

CHAPTER III

A ROUND BUOYANT JET IN A UNIFORM AMBIENT FLUID

In this chapter, theoretical solutions are obtained for an inclined round buoyant jet in a stagnant uniform environment based upon the method used previously by Fan and Brooks (12).

III-A. Formulation of the Problem

1. Conservation equations

The equation of continuity, based upon the assumed entrainment mechanism and a small variation of density, can be expressed as:

$$\frac{d}{ds} \int_0^\infty \int_0^{2\pi} u^* r dr d\varphi = 2\pi \alpha b u \quad (9)$$

the left-hand side can be integrated after substituting u^* from eq. (2). Then,

$$\frac{d}{ds} (u b^2) = 2\alpha u b \quad (\text{continuity}) \quad (10)$$

Since the pressure is assumed to be hydrostatic and there is no other force acting in the horizontal direction, the x-momentum flux should be conserved:

$$\frac{d}{ds} \int_0^\infty \int_0^{2\pi} \rho^* u^* (u^* \cos \theta) r dr d\varphi = 0 \quad (11)$$

After substituting u^* from eq. (2) and neglecting variations in ρ^* (compared to ρ^* itself):

$$\frac{d}{ds} \left(\frac{u^2 b^2}{2} \cos \theta \right) = 0 \quad (\text{x-momentum}) \quad (12)$$

In the vertical direction there is a buoyancy force acting on the jet which is equal to the rate of change of y-momentum flux:

$$\frac{d}{ds} \int_0^\infty \int_0^{2\pi} \rho^* u^* (u^* \sin \theta) r dr d\varphi = g \int_0^\infty \int_0^{2\pi} (\rho_a^* - \rho^*) r dr d\varphi \quad (13)$$

Simplifying by using both eqs. (2) and (3a), and the Boussinesq approximation, eq. (13) becomes:

$$\frac{d}{ds} \left(\frac{u^2 b^2}{2} \sin \theta \right) = g \lambda^2 b^2 \frac{\rho_0 - \rho}{\rho_0} \quad (\gamma\text{-momentum}) \quad (14)$$

The heat content or the amount of shortage of salt released from the origin must be conserved with respect to a chosen reference level in a uniform environment. By the assumption of small variation of density and linearity of temperature (or salt) - density relation, this is directly equivalent to the conservation of density deficiency flux about the reference density ρ_0 . The relation can be expressed as:

$$\frac{d}{ds} \int_0^\infty \int_0^{2\pi} u^* (\rho_0 - \rho^*) r dr d\varphi = 0 \quad (15)$$

The relation can be simplified to:

$$\frac{d}{ds} [u b^2 (\rho_0 - \rho)] = 0 \quad (\text{density deficiency}) \quad (16)$$

The relation of continuity for a certain tracer substance present in source flow only is:

$$\frac{d}{ds} \int_0^\infty \int_0^{2\pi} c^* u^* r dr d\varphi = 0 \quad (17)$$

By introducing eq. (4) it can be written as:

$$\frac{d}{ds} (c u b^2) = 0 \quad (\text{continuity of tracer}) \quad (18)$$

2. Geometric relations

To determine the jet trajectory the following geometric equations must be solved simultaneously with the previous set of equations:

$$\frac{dx}{ds} = \cos \theta \quad (\text{geometry}) \quad (19)$$

and,

$$\frac{dy}{ds} = \sin \theta \quad (\text{geometry}) \quad (20)$$

Therefore, the problem has seven unknowns, namely, u , b , $(\rho_0 - \rho)$, θ , c , x and y to be solved from seven equations, i. e. eqs. (10), (12), (14), (16), (18), (19), and (20).

3. Initial conditions

The initial conditions given at the origin for this system of ordinary differential equations are:

$$u(0) = U_0, \quad b(0) = b_0, \quad \rho(0) = \rho_1, \quad \theta(0) = \theta_0, \quad (21)$$

$$c(0) = c_0, \quad x = 0 \text{ and } y = 0 \text{ at } s = 0$$

Eqs. (12), (16) and (18) can be integrated immediately. The equation of x-momentum is:

$$\frac{u^2 b^2}{2} \cos \theta = \text{const.} \quad (\text{x-momentum}) \quad (22)$$

The equation of density deficiency (16) is:

$$ub^2(\rho_0 - \rho) = \text{const.} = U_0 b_0^2 (\rho_0 - \rho_1) \quad (\text{density deficiency}) \quad (23)$$

The equation of tracer concentration is simply:

$$cub^2 = \text{const.} = c_o U_o b_o^2$$

(continuity of a trace) (24)

From eqs. (23) and (24), $(\rho_o - \rho_1)$ and c are directly related to the solutions of u and b . Thus these two equations will not be carried through the following calculation.

III-B. Normalized Equations and Dimensionless Parameters

In order to transform the system of equations into a simple normalized form, dimensionless parameters are defined as follows:

Volume flux parameter:

$$\mu = ub^2 / (U_o b_o^2) \quad (25)$$

Momentum flux parameters:

$$m = \left\{ \frac{g \lambda^2 U_o^3 b_o^6 (\rho_o - \rho_1)}{4 \sqrt{2} \alpha \rho_o} \right\}^{-2/5} \frac{u^2 b^2}{2} \quad (26)$$

$$h = m \cos \theta \quad (27)$$

$$v = m \sin \theta \quad (28)$$

Coordinates:

$$s: \quad \zeta = \left\{ \frac{\rho_o U_o^2 b_o^4}{32 \alpha^4 \lambda^2 g (\rho_o - \rho_1)} \right\}^{-1/5} s \quad (29)$$

$$x: \quad \eta = \left\{ \frac{\rho_o U_o^2 b_o^4}{32 \alpha^4 \lambda^2 g (\rho_o - \rho_1)} \right\}^{-1/5} x \quad (30)$$

$$y: \quad \xi = \left\{ \frac{\rho_o U_o^2 b_o^4}{32 \alpha^4 \lambda^2 g (\rho_o - \rho_1)} \right\}^{-1/5} y \quad (31)$$

Equations (10), (12), (14), (19) and (20) then become:

$$\frac{d\mu}{d\zeta} = \sqrt{m} \quad (32)$$

$$h = \sqrt{m^2 - v^2} = \text{constant} = h_0 \quad (33)$$

$$\frac{dv}{d\zeta} = \frac{\mu}{m} \quad (34)$$

$$\frac{d\eta}{d\zeta} = \frac{h}{m} \quad (35)$$

$$\frac{d\xi}{d\zeta} = \frac{v}{m} \quad (36)$$

The corresponding initial conditions (21) are:

$$\mu(0) = 1, m(0) = m_0, \theta(0) = \theta_0, \eta = 0, \text{ and } \xi = 0 \text{ at } \zeta = 0 \quad (37)$$

The solutions to the system of differential equations cannot be obtained in closed analytical form. Thus numerical integrations were carried out.

The numerical solutions were obtained by direct step-by-step integration on an IBM 7094 digital computer using a subroutine "DEQ/DIFFERENTIAL EQUATION SOLVER" at Booth Computing Center of California Institute of Technology. The subroutine was based upon Runge-Kutta-Gill method with automatic control of truncation error. The same numerical method was used in solving all the other classes of problems analyzed in this report.

There are two relevant parameters for this problem, namely θ_0 and m_0 . For each combination of these two parameters, a computer run was made until the value $\xi\sqrt{m_0}$ reached 50. A wide range of θ_0 and m_0 values were covered in the following section.

III-C. Solutions of Jet Gross Behavior

Numerical solutions were carried out for cases of various angles of discharge: $\theta_0 = 0^\circ$ (horizontal discharge), 15° , 30° , 45° , 60° and 90° (vertical discharge) over a range of $m_0 = 0.37$ to 18. Figs. 4(a) to (d) are graphs of jet trajectories ($\xi\sqrt{m_0}$ vs. $\eta\sqrt{m_0}$) for $\theta_0 = 90^\circ$; only b/b_0 values are plotted as shown in Fig. 4(e) since in this case jet trajectories are all vertical. The variables $\xi\sqrt{m_0}$ and $\eta\sqrt{m_0}$ are convenient because of the following relationships:

$$\frac{x}{b_0} = \frac{\eta\sqrt{m_0}}{2\alpha}; \quad \frac{y}{b_0} = \frac{\xi\sqrt{m_0}}{2\alpha}. \quad (37a)$$

As shown in these figures, the initial angle of discharge θ_0 has an important effect on jet trajectories. The initial momentum flux parameter m_0 affects the jet trajectories also but its effect diminishes as θ_0 tends to 90° . b/b_0 tends to increase with m_0 as shown in Fig. 4(e).

The jet centerline dilution ratio S_0 with respect to the concentration value at O, i.e. at the end of the zone of flow establishment, from eqs. (24) and (25), is:

$$S_0 = c_0/c = \mu \quad (37b)$$

S_0 values are plotted as functions of $\xi\sqrt{m_0}$ and m_0 for various θ_0 cases as shown in Figs. 5(a) to (f). It is interesting to notice that for non-vertical cases, there are regions of minimum dilution ratios at some $\xi\sqrt{m_0}$ and m_0 values. It is mainly due to jet geometry. The reason is as follows: at small m_0 values, the jet rises vertically as a simple plume, $S_0 \sim (\xi\sqrt{m_0})^{5/3}$. The resulting S_0 curves are as shown in Fig. 5(f) for $\theta_0 = 90^\circ$. But a non-vertical jet travels a much longer path at large m_0 before reaching a fixed vertical distance $\xi\sqrt{m_0}$ than the jet with small m_0 .

Both Figs. 4 and 5 will be useful in establishing gross jet behavior. Dilution values at $\xi\sqrt{m_0}$ greater than 50 can be obtained by utilizing the simple round plume relation, $S_0 \approx 0.46 \xi^{5/3}$ in small and intermediate m_0 cases.

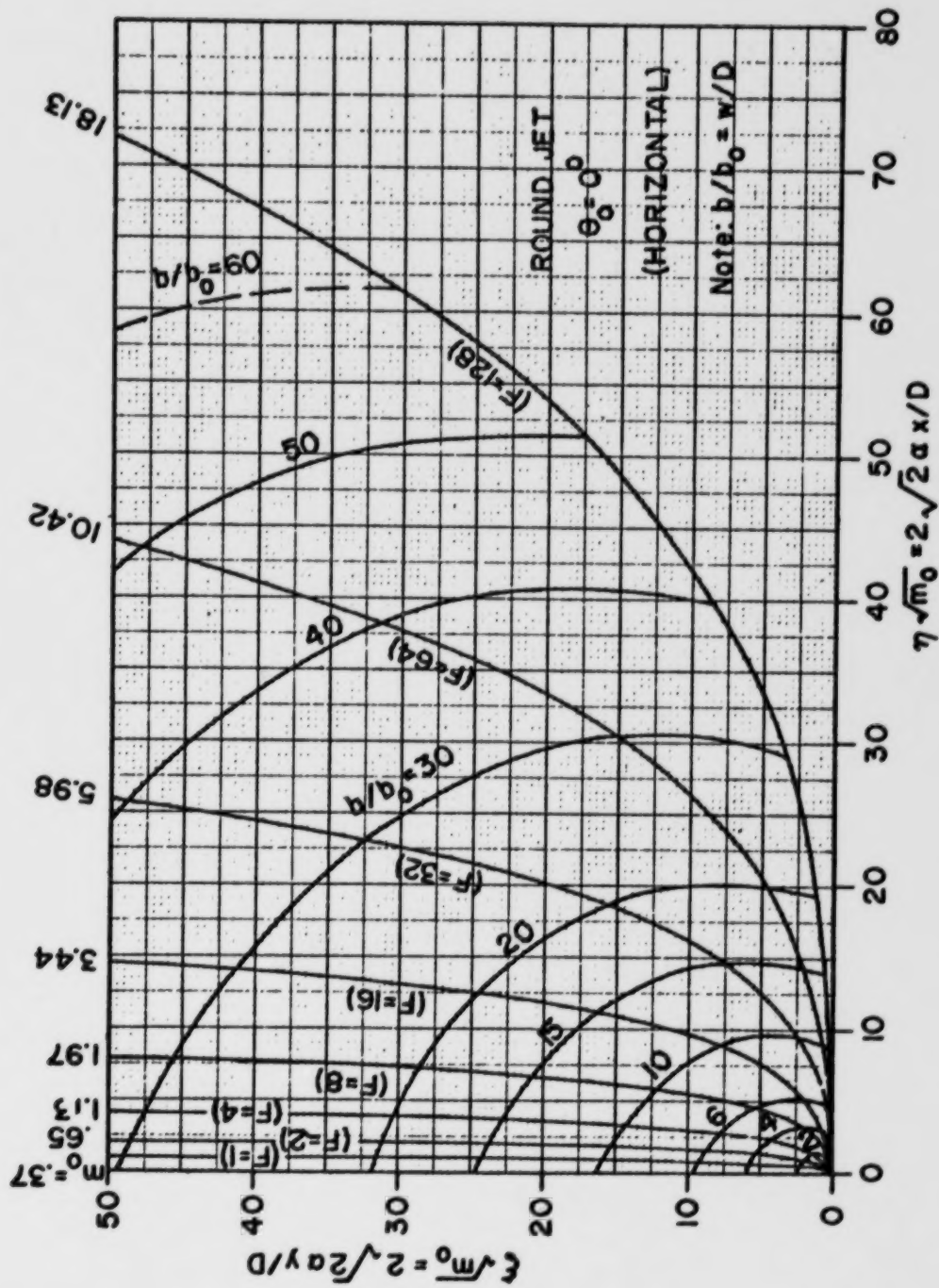


Fig. 4(a). Trajectories and Half-width b/b_0 of Round Buoyant Jets in Stagnant Uniform Ambient Fluids: $\theta_0 = 0^\circ$. (Froude numbers (F) shown are based on $\alpha = 0.082$.)

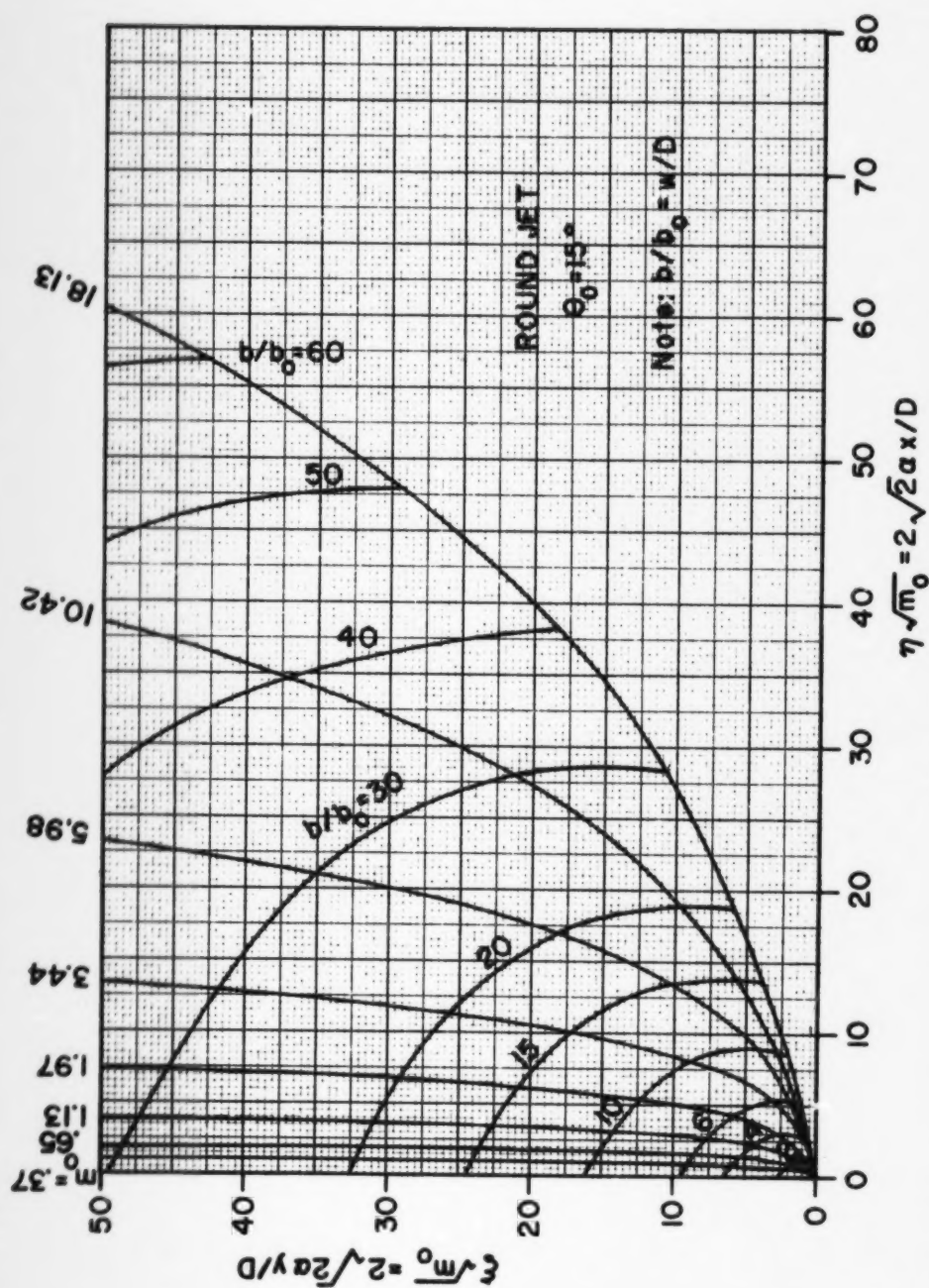


Fig. 4(b). Trajectories and Half-width b/b_0 of Round Buoyant Jets in Stagnant Uniform Ambient Fluids: $\theta_0 = 15^\circ$.

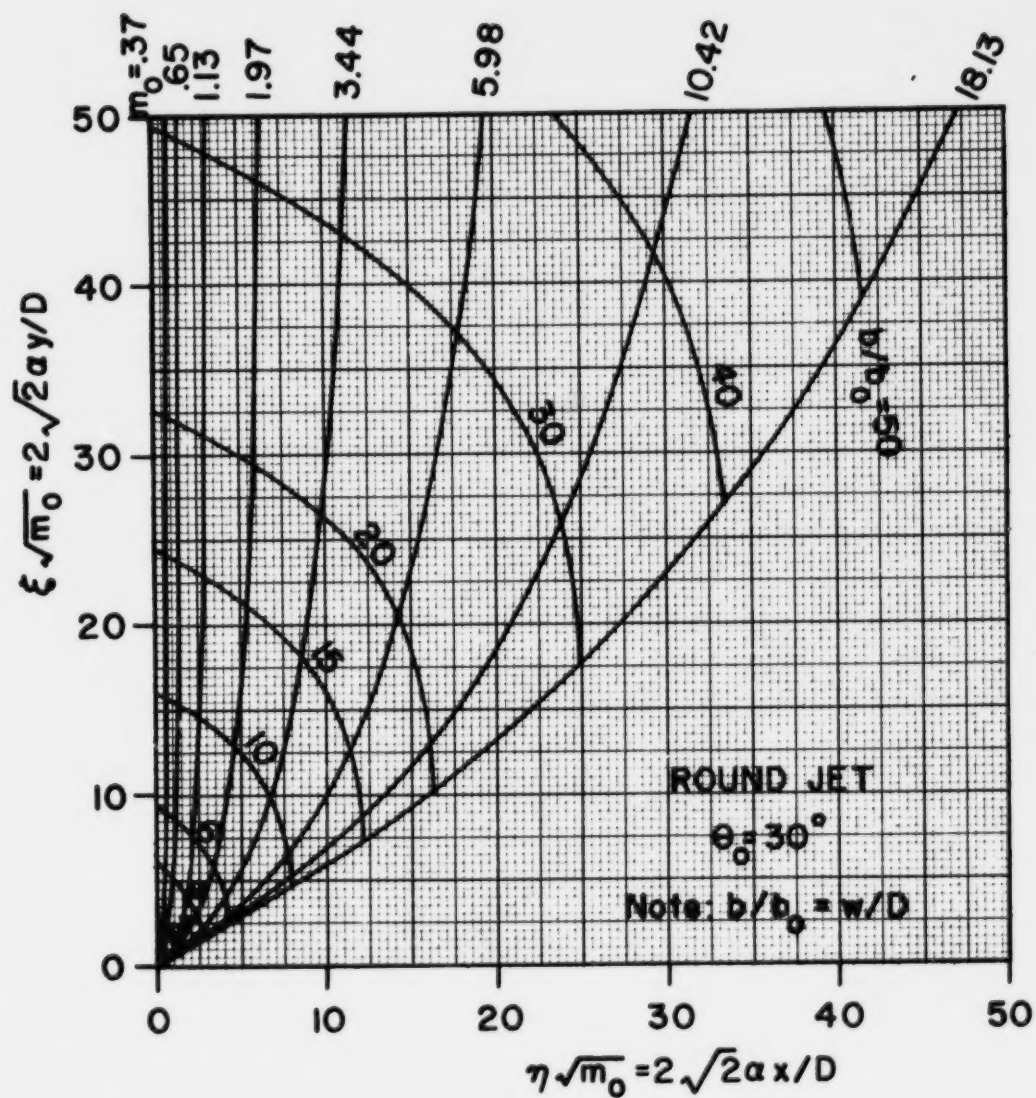


Fig. 4(c). Trajectories and Half-Width b/b_0 of Round Buoyant Jets in Stagnant Uniform Ambient Fluids: $\theta_0 = 30^\circ$.

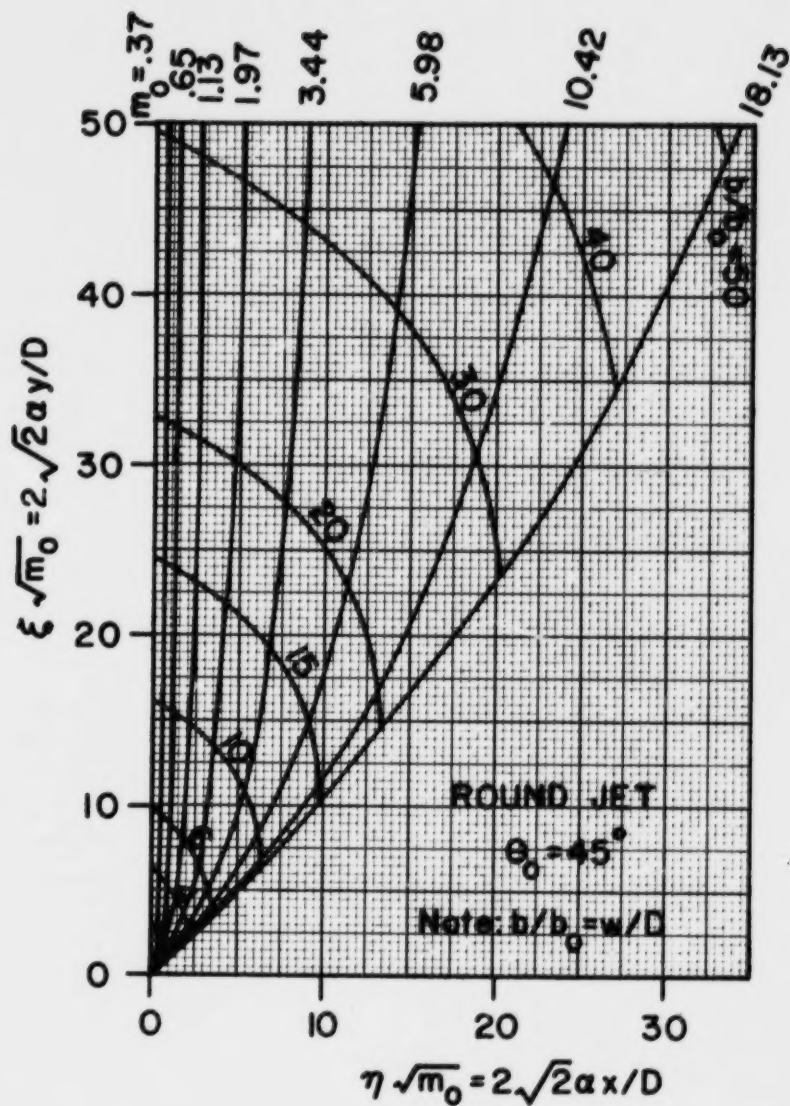


Fig. 4(d). Trajectories and Half-Width b/b_0 of Round Buoyant Jets in Stagnant Uniform Ambient Fluids: $\theta_0 = 45^\circ$.

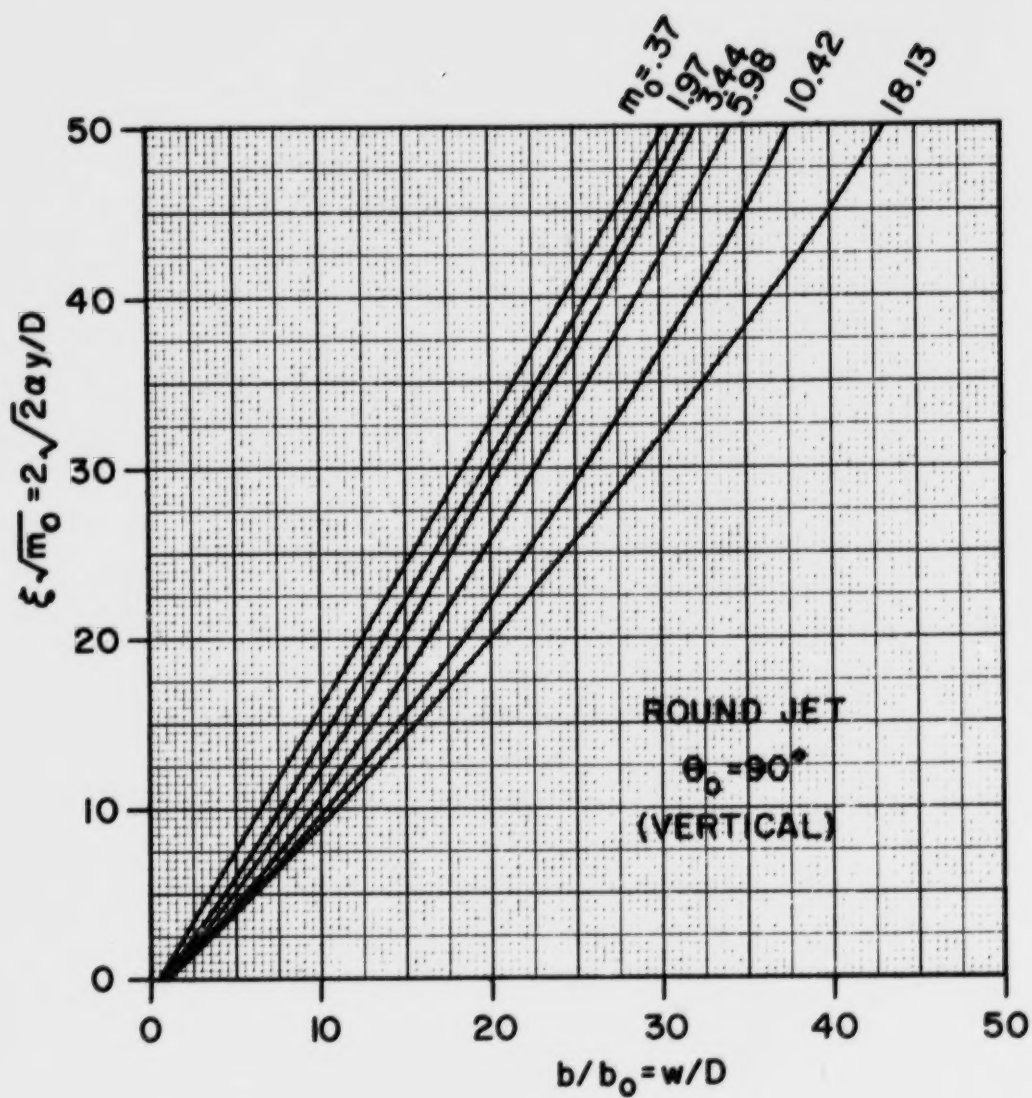


Fig. 4(e). Half-Width b/b_0 of Round Buoyant Jets in Stagnant Uniform Ambient Fluids: $\theta_0 = 90^\circ$. (Trajectories are all vertical lines.)

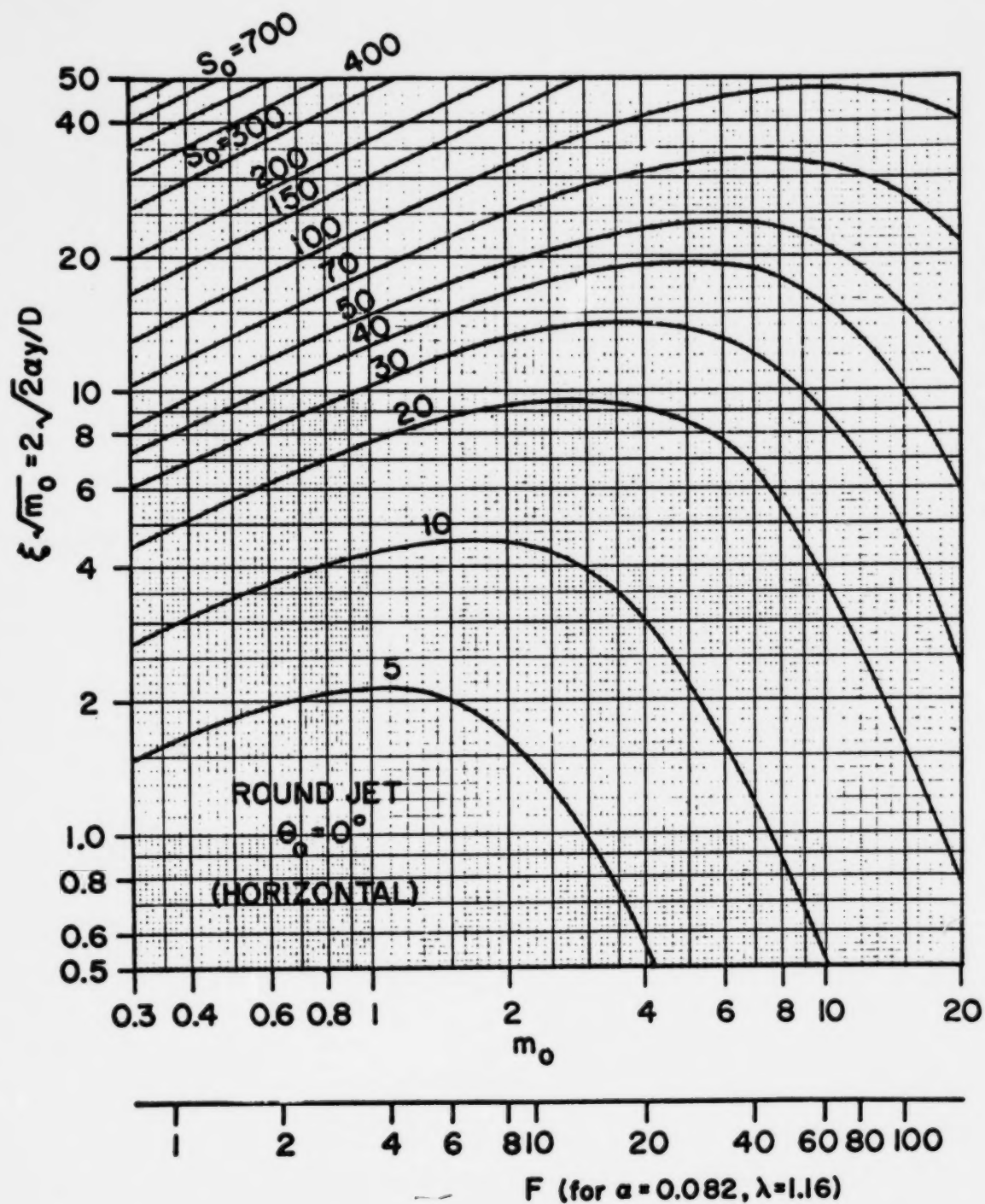


Fig. 5(a). Dilution of Round Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 0^\circ$.

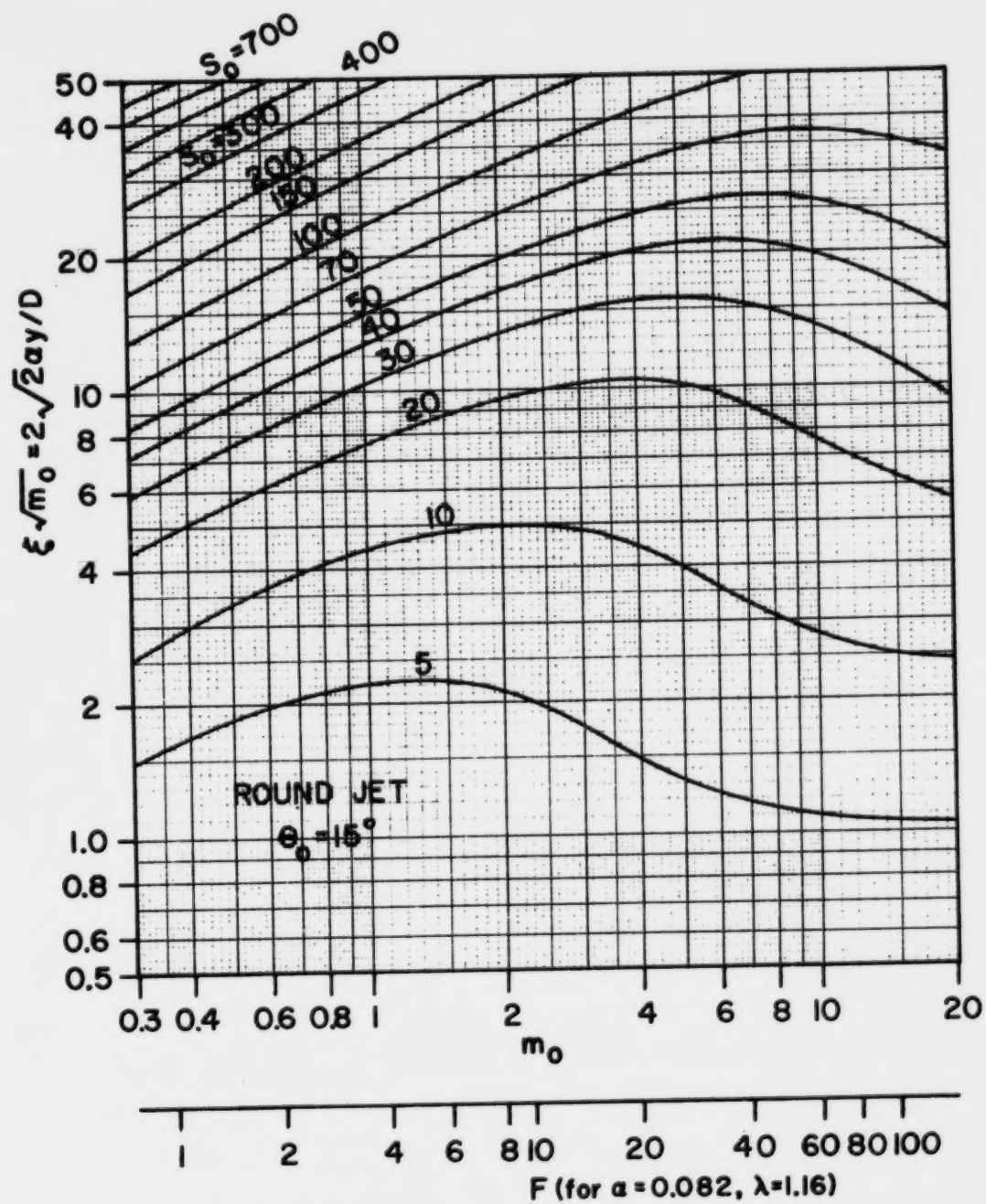


Fig. 5(b). Dilution of Round Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 15^\circ$.

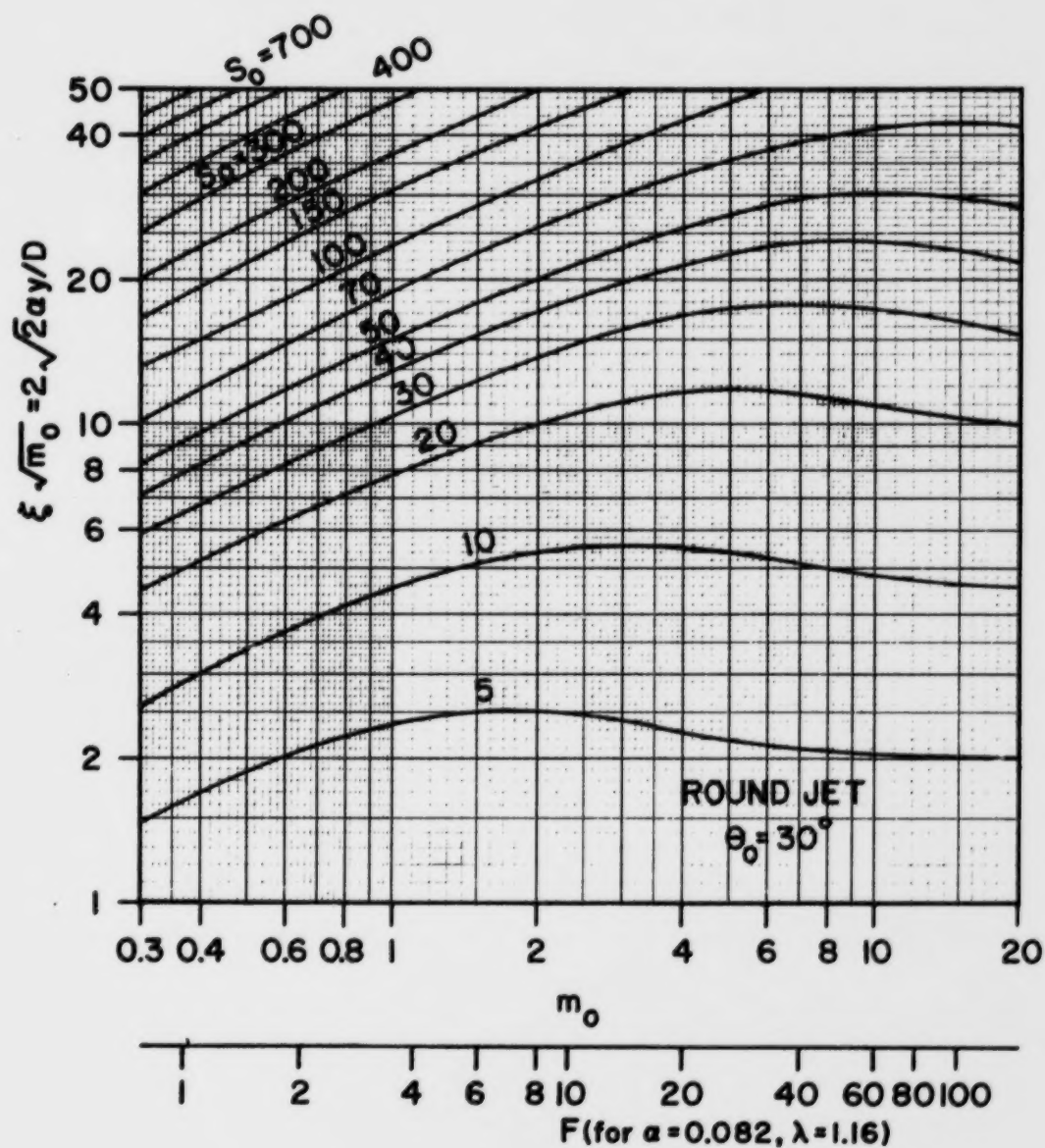


Fig. 5(c). Dilution of Round Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 30^\circ$.

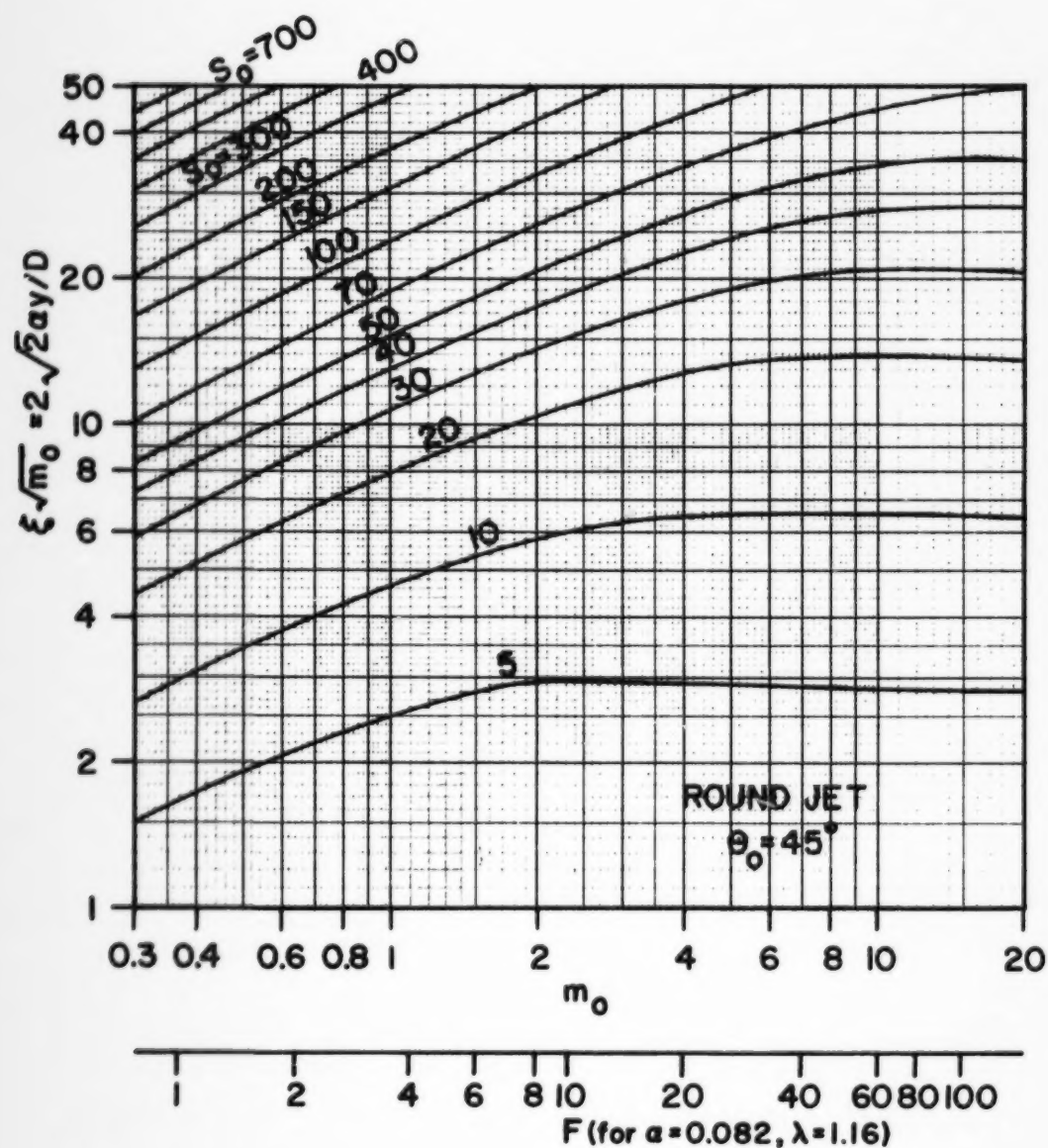


Fig. 5(d). Dilution of Round Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 45^\circ$.

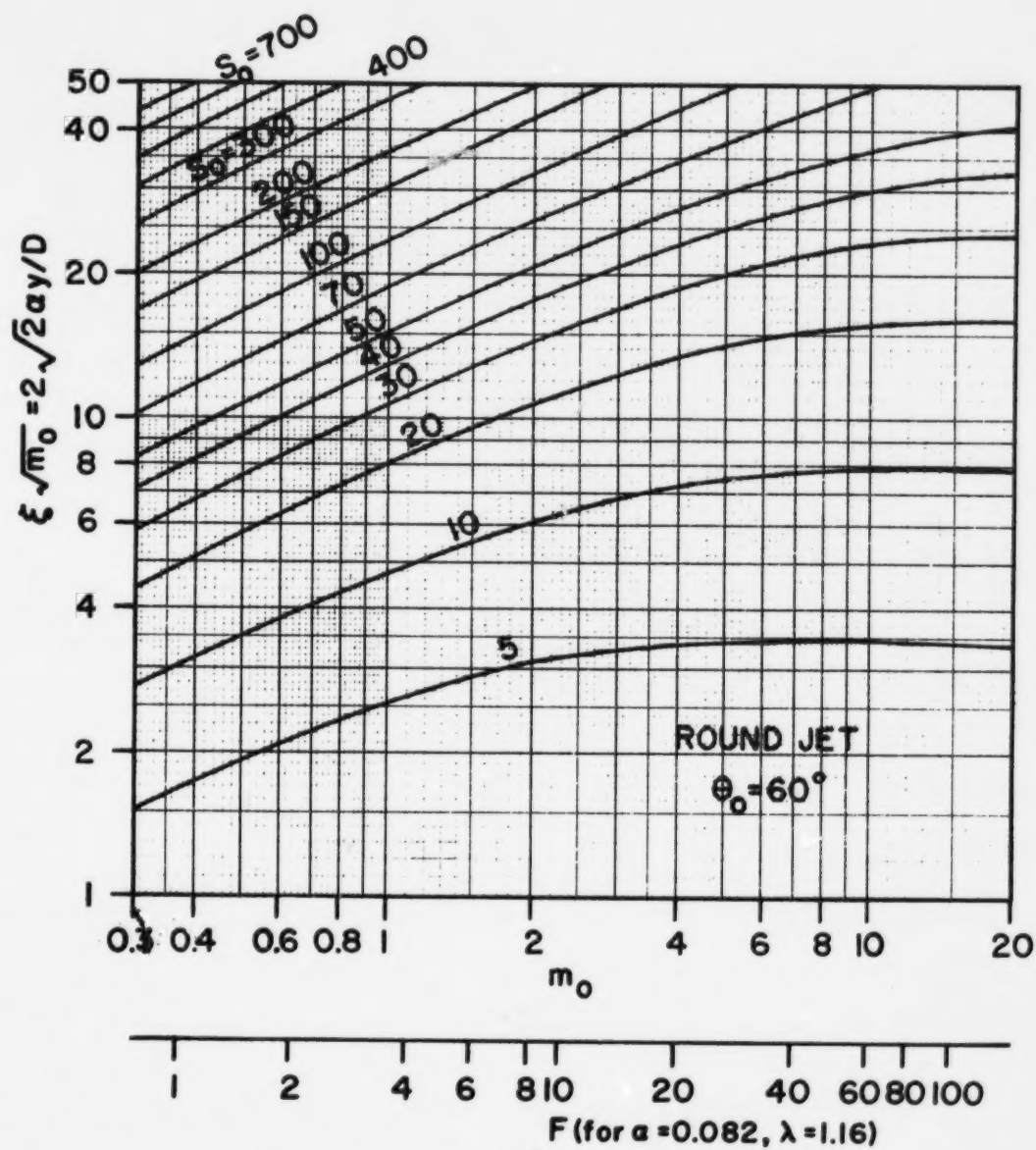


Fig. 5(e). Dilution of Round Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 60^\circ$.

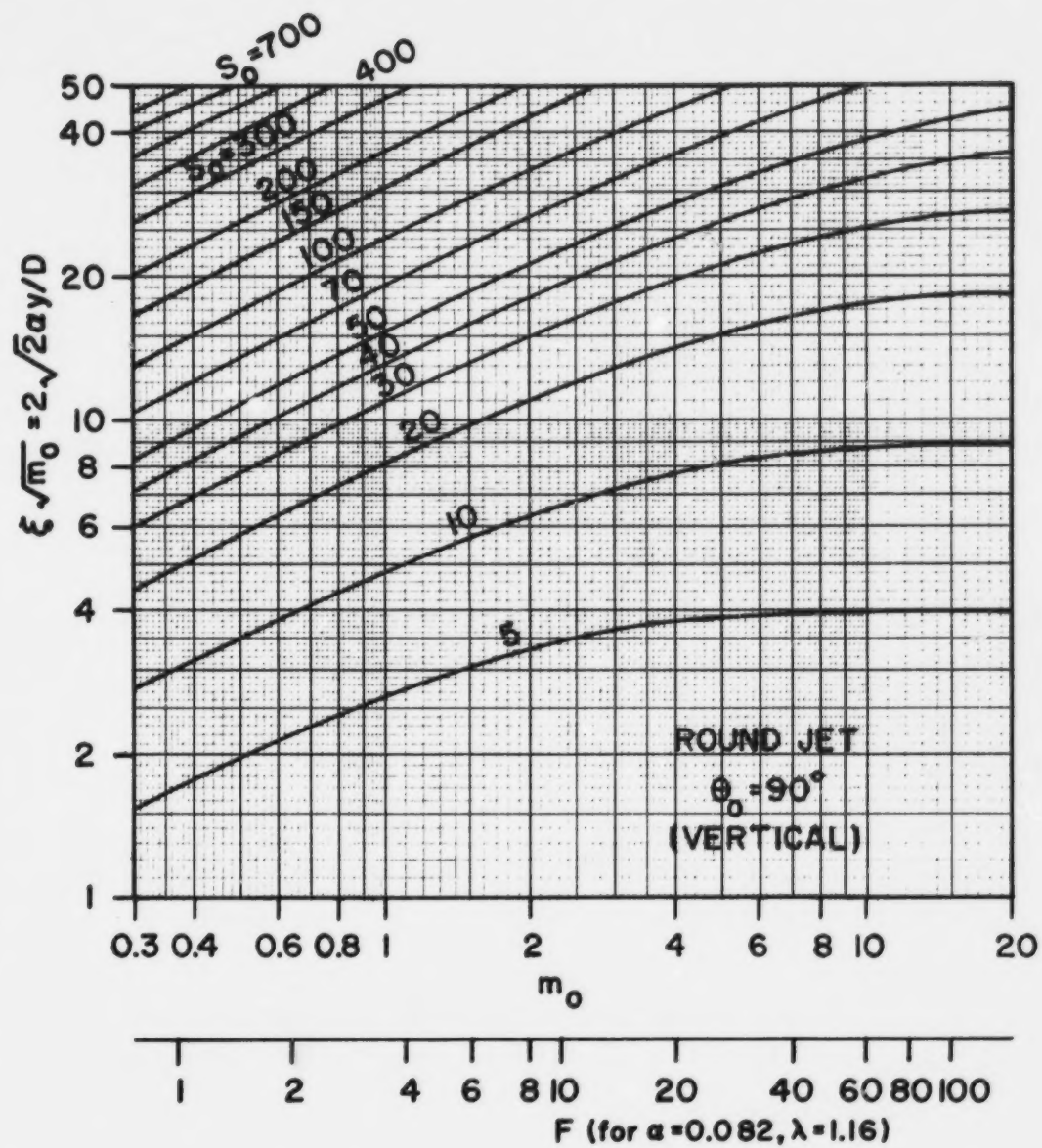


Fig. 5(f). Dilution of Round Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 90^\circ$.

III-D. Application to Practical Problems

For application to practical problems where the jet is issued from a nozzle or orifice, it is necessary to consider the zone of flow establishment. Assuming that the results of Albertson et al. (5) apply also for the zone of flow establishment of an inclined round buoyant jet, the length of the zone is taken to be $6.2D$ where D is the initial jet diameter as shown in Fig. 6. The initial top-hat profiles at O' develop into Gaussian distributions given by eqs. (6), (7a), and (8) at the end of the zone.

The initial value b_o can be obtained by applying the momentum relation between cross sections at O and O' assuming that the buoyancy force is negligible in such a short region:

$$\begin{aligned} \frac{\pi}{4} D^2 U_o^3 &= \int_{A \text{ at } O} u^{*2} dA \\ &= \frac{\pi b_o^2 U_o^3}{2} \end{aligned} \quad (38)$$

thus,

$$b_o = \frac{D}{\sqrt{2}} \quad (39)$$

The trace concentrations c_o at O , assuming that the concentration at O' is unity, can be obtained by considering the continuity relation for the tracer:

$$\begin{aligned} 1 \cdot \frac{\pi}{4} D^2 U_o &= \int_{A \text{ at } O} u^* c^* dA \\ &= \frac{\lambda^2}{1+\lambda^2} \pi b_o^2 U_o c_o \end{aligned} \quad (40)$$

thus,

$$c_o = \frac{1+\lambda^2}{2\lambda^2} \quad (41)$$

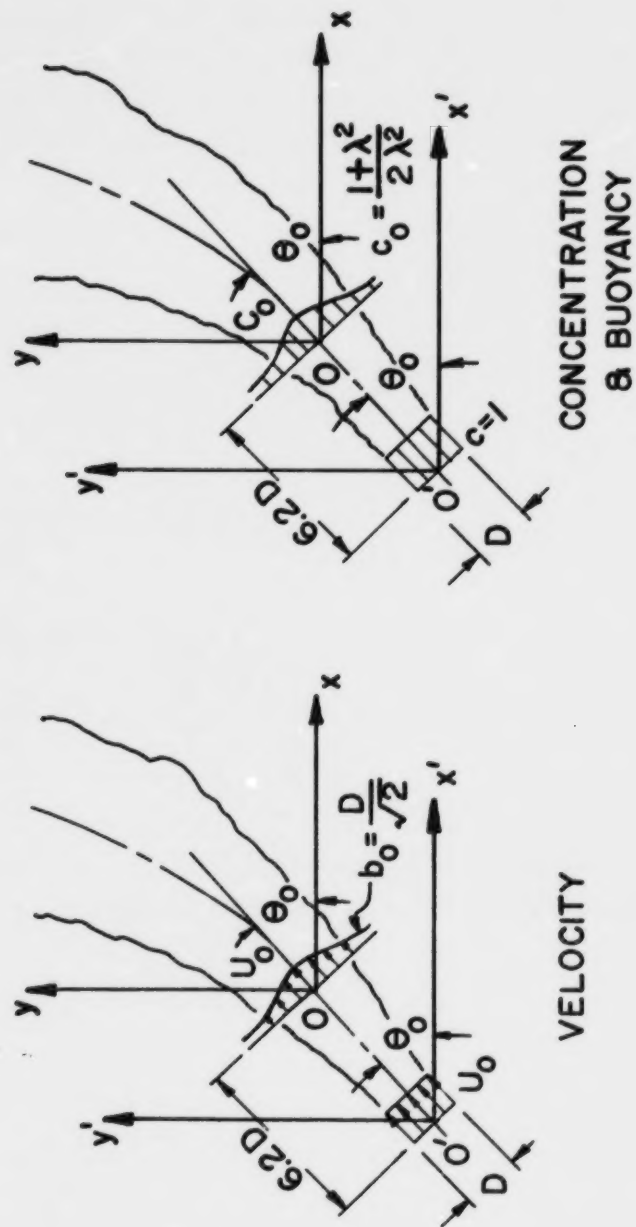


Fig. 6. Zone of Flow Establishment for an Inclined Round Buoyant Jet.

The dilution ratio S with respect to the initial concentration at the discharge point O' is thus:

$$S = \frac{2\lambda^2}{1+\lambda^2} S_o = \frac{2\lambda^2}{1+\lambda^2} \mu \quad (42)$$

Hence the values of S_o given in Fig. 5 must be multiplied by the factor $2\lambda^2/(1+\lambda^2)$ to obtain the true centerline dilution S with respect to the initial value at the nozzle. These above relations are also adopted later in Chapter IV for round buoyant jets in a stagnant environment with linear density stratification.

If a densimetric jet Froude number F is defined as:

$$F = \frac{U_o}{\sqrt{\frac{\rho_o - \rho_1}{\rho_o} gD}} \quad (43)$$

It can be shown that F is directly related to m_o by:

$$F = \frac{\lambda}{2^{\frac{1}{4}} \sqrt{\alpha}} m_o^{5/4}, \text{ or } m_o = \left(\frac{2\alpha^2}{\lambda^4} \right)^{1/5} F^{4/5} \quad (44)$$

Thus the problem of an inclined round buoyant jet in a stagnant uniform environment is characterized by two parameters, namely either F and θ_o , or m_o and θ_o . The dimensionless distances x/D and y/D are related to η and ξ .

$$x/D = \eta \sqrt{m_o} / (2\sqrt{2}\alpha) \quad (45)$$

and

$$y/D = \xi \sqrt{m_o} / (2\sqrt{2}\alpha) \quad (46)$$

Thus $\xi\sqrt{m_o}$ and $\eta\sqrt{m_o}$ are chosen for dimensionless representation of jet trajectories as shown in Figs. 4(a) to (d).

The dimensionless distances with respect to the discharge point O' , i. e. x'/D and y'/D , are:

$$x'/D = x/D + 6.2 \cos \theta_0 \quad (47)$$

and

$$y'/D = y/D + 6.2 \sin \theta_0 \quad (48)$$

Suggested values for the entrainment coefficient α and turbulent Schmidt number λ^2 are:

$$\alpha = 0.082 \text{ and } \lambda = 1.16$$

based upon Rouse, Yih and Humphreys' results (23). These values were adopted by numerous investigators as discussed in detail by Fan (10), and Fan and Brooks (12). Using these values, eqs. (42), (44) to (46) become:

$$S = 1.15 S_0 = 1.15 \mu \quad (42a)$$

$$F = 3.40 m_0^{5/4}, \text{ or } m_0 = 0.374 F^{4/5} \quad (44a)$$

$$x/D = 4.32 \eta \sqrt{m_0} \quad (45a)$$

$$y/D = 4.32 \xi \sqrt{m_0} \quad (46a)$$

For simple round jets ($m_0 = \infty$, $F = \infty$), the best choice of σ is 0.057 based upon Albertson et al. (5).

CHAPTER IV

A ROUND BUOYANT JET IN A LINEARLY DENSITY-STRATIFIED AMBIENT FLUID

In this chapter, theoretical solutions are given for inclined round buoyant jets in a stagnant environment with linear density gradient. The analysis is given previously by Fan (10), but more numerical results are incorporated here. Many equations are in common with those presented in the previous chapter for round jets in a uniform environment.

IV-A. Formulation of the Problem

1. Conservation equations

The basic conservation equations are the same as eqs. (9), (11), (13) and (17). For a round jet in a stagnant environment with linear density stratification eq. (9) reduces to eq. (10), eq. (11) to eq. (12), and eq. (17) to eq. (18) without change. However, with the ambient density ρ_a varying with y instead of being constant, eq. (13) becomes

$$\frac{d}{ds} \left(\frac{u^2 b^2}{2} \sin \theta \right) = g \lambda^2 b^2 \frac{\rho_a - \rho_0}{\rho_0} \quad (\text{y-momentum}) \quad (49)$$

Furthermore the buoyancy flux (relative to the reference density ρ_0) is no longer conserved because of the entrainment along the buoyant jet; eq. (15) must now be written:

$$\frac{d}{ds} \int_0^\infty \int_0^{2\pi} u^* (\rho_0 - \rho_a^*) r dr d\varphi = \alpha u b \int_0^{2\pi} (\rho_0 - \rho_a^*(s, b, \varphi)) d\varphi \quad (50a)$$

The last expression implies that the ambient density ρ_a^* varies around the edge of the buoyant plume. But for linear stratification, we may use ρ_a^* as just $\rho_a(y)$, and after some manipulations the resulting equation for density deficiency flux is:

$$\frac{d}{ds} [u b^2 (\rho_a - \rho)] = \frac{1+\lambda^2}{\lambda^2} b^2 u \frac{d\rho_a}{ds} \quad (\text{density deficiency}) \quad (50b)$$

2. Geometric relations

The jet trajectory is again expressed by eqs. (19) and (20). Thus, the problem has seven unknowns, namely, u , b , $\rho_a - \rho$, θ , c , x , and y , to be solved from seven equations, i. e., eqs. (10), (12), (49), (50b), (18), (19) and (20).

3. Initial conditions

The initial conditions are the same as represented by eq. (21). The equations of x -momentum and tracer concentration can be integrated as eqs. (22) and (24) and the latter will not be carried further in the analysis.

IV-B. Normalized Equations and Dimensionless Parameters

In order to transform the system of equations into simple normalized form, dimensionless parameters are defined as follows:

Volume flux parameter:

$$u = \left\{ G^5 / F_o^6 64 \alpha^4 (1+\lambda^2) \right\}^{1/8} u b^2 \quad (51)$$

Momentum flux parameters:

$$\begin{aligned} m &= \left\{ G / [(1+\lambda^2) F_o^2] \right\} b^4 u^4 / 4 && (\text{in } s\text{-direction}) \\ h &= m \cos^2 \theta && (\text{in } x\text{-direction}) \\ v &= m \sin^2 \theta && (\text{in } y\text{-direction}) \end{aligned} \quad (52)$$

Buoyancy flux parameter:

$$\theta = \left\{ \frac{\lambda^2}{1+\lambda^2} b^2 u g \frac{\rho_a - \rho}{\rho_o} \right\} / F_o \quad (53)$$

Coordinates:

$$\begin{aligned} s: \quad \zeta &= \left\{ G^3 64 \alpha^4 (1 + \lambda^2) / F_0^2 \right\}^{1/8} s \\ x: \quad \eta &= \left\{ G^3 64 \alpha^4 (1 + \lambda^2) / F_0^2 \right\}^{1/8} x \\ y: \quad \xi &= \left\{ G^3 64 \alpha^4 (1 + \lambda^2) / F_0^2 \right\}^{1/8} y \end{aligned} \quad (54)$$

where the parameters $G = -\frac{g}{\rho_0} \frac{d\rho_a}{dy}$ and $F_0 = \frac{\lambda^2}{1 + \lambda^2} b_0^2 U_0 g \frac{\rho_0 - \rho_1}{\rho_0}$ are not dimensionless.

Equations (10), (22), (49), (50b), (19), and (20) then become respectively:

$$d\mu/d\zeta = m^{\frac{1}{2}} \quad (55)$$

$$h = m - v = h_0 = \text{const.} \quad (56)$$

$$dv/d\zeta = \beta \mu (v/m)^{\frac{1}{2}} \quad (57)$$

$$d\beta/d\zeta = -\mu (v/m)^{\frac{1}{2}} \quad (58)$$

$$d\eta/d\zeta = (h/m)^{\frac{1}{2}} \quad (59)$$

$$d\xi/d\zeta = (v/m)^{\frac{1}{2}} \quad (60)$$

The corresponding initial conditions (21) are:

$$\mu(0) = \mu_0, \quad m(0) = m_0, \quad \eta(0) = \eta_0, \quad \xi(0) = 1,$$

$$\eta = 0 \text{ and } \xi = 0 \text{ at } \zeta = 0 \quad (61)$$

The solutions to the system of differential equations cannot be obtained in closed analytical form. Thus numerical integrations were performed.

The numerical solutions were obtained by a method similar to that described in Section III-B. There are three relevant parameters for the problem, namely μ_0 , m_0 and θ_0 . For each combination of these parameters a computer run was made. The integration stopped at the terminal point of the jet. A substantial range of initial conditions was covered by the numerical computation. These results are presented in the following section.

IV-C. Solution of Jet Gross Behavior

Fig. 7 shows the variations of gross jet characteristics v , β and μ along the ζ -coordinate for the case $m_0 = 0.2$, $\mu_0 = 0$ and $\theta_0 = 0$. The vertical momentum flux parameter v first increases with ζ and reaches its maximum at the point where β changes sign and then decreases under negative buoyancy and finally vanishes at the terminal point ζ_t . The buoyancy flux parameter β decreases monotonically from unity to a minimum at ζ_t . The volume flux parameter μ increases with ζ and reaches μ_t at ζ_t .

The effect of initial angle of discharge θ_0 on the jet trajectory is demonstrated in Figs. 8 and 9. For $m_0 = 0.2$ and 2.0 with $\mu_0 = 0$, trajectories were obtained for different θ_0 values. The terminal height of rise ξ_t increases with increasing θ_0 .

The variation of the terminal height of rise ξ_t for the range $\mu_0 = 0$ to 0.01 is shown in Fig. 10. The curve for $m_0 = 2.0$ shows a faster rate of decrease in ξ_t as the angle θ_0 decreases than the cases for smaller m_0 values. For a simple plume where m_0 vanishes the height is independent of the initial angle of discharge. Fig. 11 shows the variation of the terminal volume flux parameter μ_t over the same range of initial conditions. A substantial increase in μ_t can be achieved by decreasing the θ_0 at large m_0 values. Both Figs. 10 and 11 demonstrate that the variation of the terminal quantities ξ_t and μ_t are almost independent of the initial volume flux parameter μ_0 in the range to 0.01 . Since the centerline dilution ratio S_0 at the terminal point is simply the ratio of the terminal to the initial volume flux parameters, an increase in the terminal dilution ratio can be obtained by decreasing μ_0 .

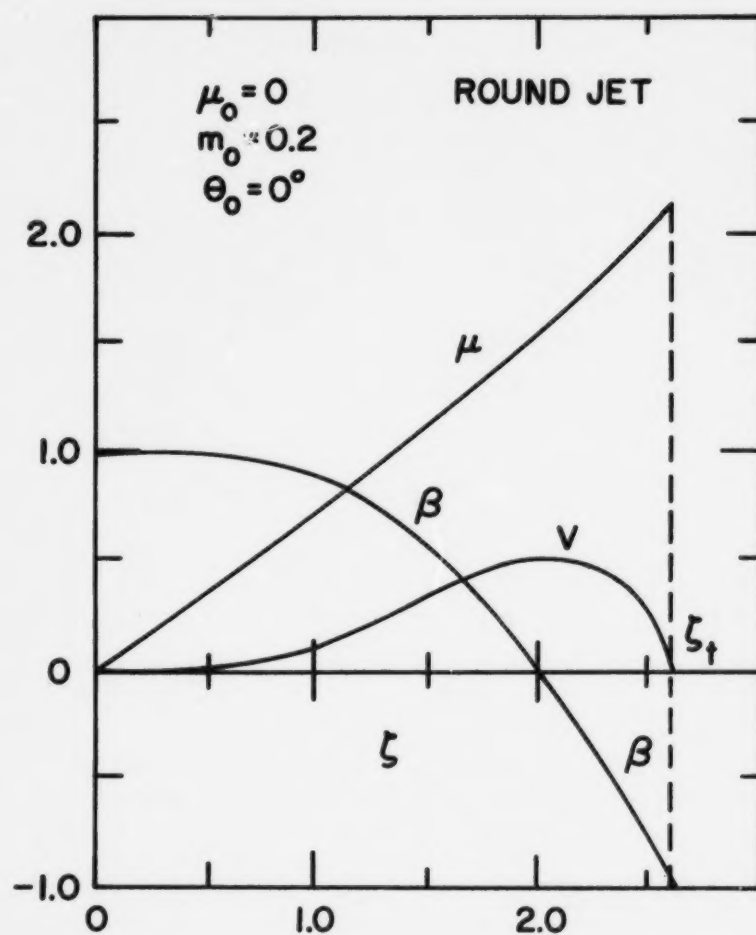


Fig. 7. Variation of Volume Flux Parameter μ , Buoyancy Flux Parameter β and Vertical Momentum Flux Parameter v along ζ - Coordinate for a Horizontal Buoyant Jet with $\mu_0 = 0$ and $m_0 = 0.2$.

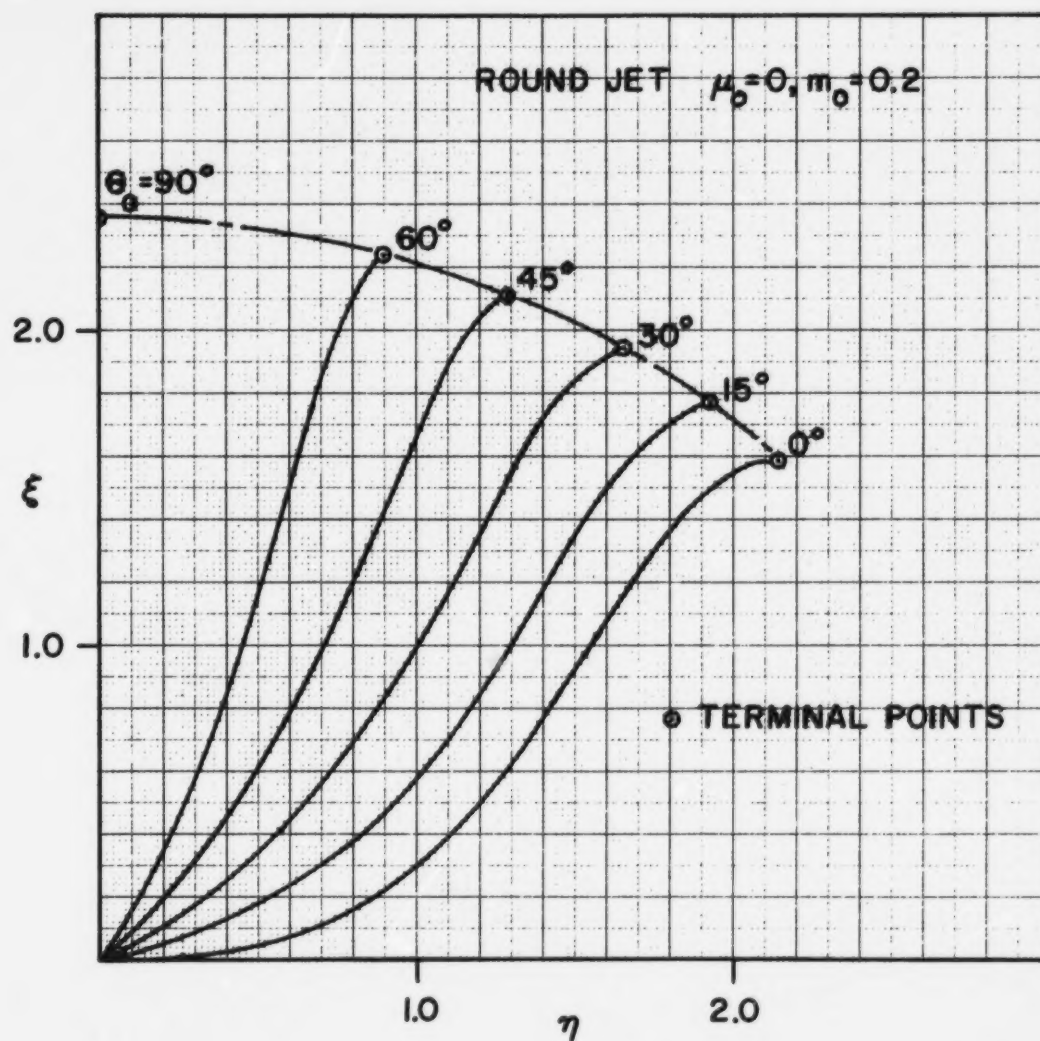


Fig. 8. Paths of Inclined Round Buoyant Jets with $\mu_0 = 0, m_0 = 0.2$.

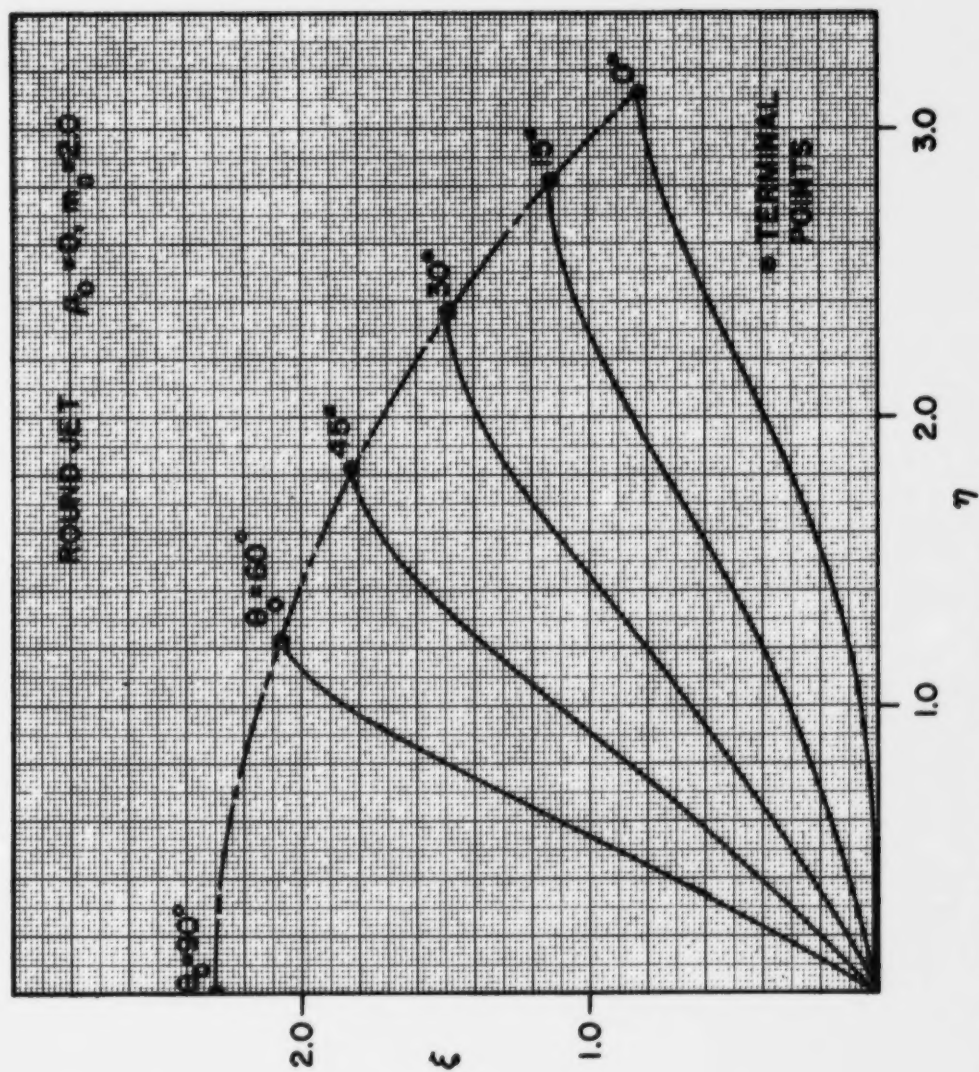


Fig. 9. Paths of Inclined Round Buoyant Jets with $u_0 = 0, m_0 = 2.0$.

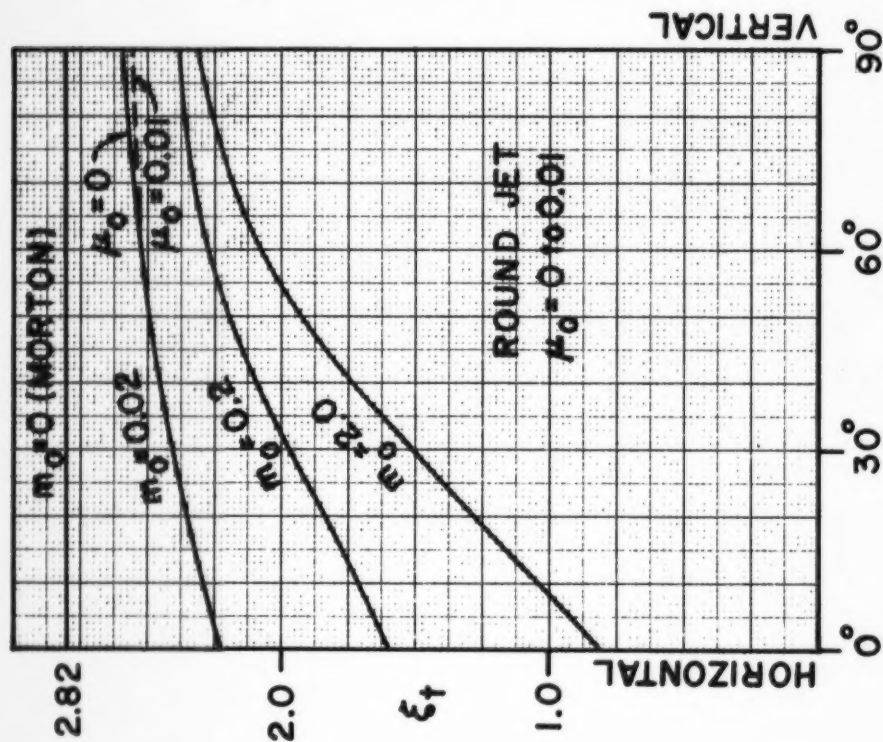


Fig. 10. Terminal Height of Rise ξ_t for Inclined Round Buoyant Jets with $\mu_o = 0$ to 0.01.

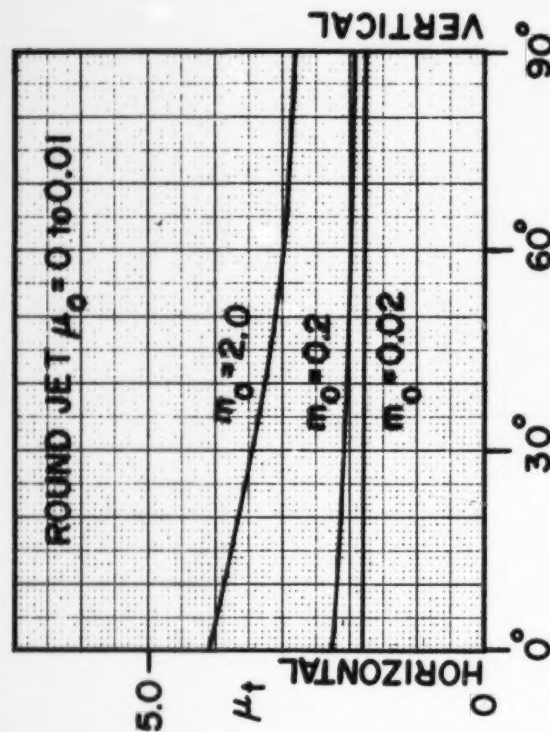


Fig. 11. Terminal Volume Flux Parameter μ_t for Inclined Round Buoyant Jets with $\mu_o = 0.01$. (Note: The terminal dilutions ratio $S = \mu_t/\mu_o$.)

Numerical solutions for both vertical and horizontal buoyant jets were carried out covering a wide range of initial conditions. The variations of ξ_t and μ_t values for vertical jets ($\theta_0 = 90^\circ$) are shown in Figs. 12 and 13. The variations of ξ_t , η_t and μ_t values for horizontal jets ($\theta_0 = 0^\circ$) are shown in Figs. 14, 15 and 16.

IV-D. Application to Practical Problems

For application to a practical problem, the zone of flow establishment is again assumed to follow Albertson et al.'s (5) results as indicated in Section III-D (Fig. 6). Eqs. (38) to (41), (47) and (48) are assumed to be equally applicable to a round buoyant jet in a linearly stratified environment.

The dilution ratio with respect to the initial concentration at the discharge point O' is:

$$S = \frac{2\lambda^2}{1+\lambda^2} \quad S_o = \frac{2\lambda^2}{1+\lambda^2} \frac{\mu}{\mu_o} \quad (62)$$

If a densimetric jet Froude number F is defined as:

$$F = \frac{U_o}{\sqrt{\frac{\rho_o - \rho_1}{\rho_o} gD}} \quad (63)$$

and a stratification parameter T as:

$$T = \frac{\rho_o - \rho_1}{D(-\frac{d\rho_a}{dy})} \quad (64)$$

It can be shown that the problem of an inclined buoyant jet in a linearly stratified environment is characterized by three parameters, namely F, T and θ_0 . The relationships between the set of parameters F and T and the set m_o , μ_o are:

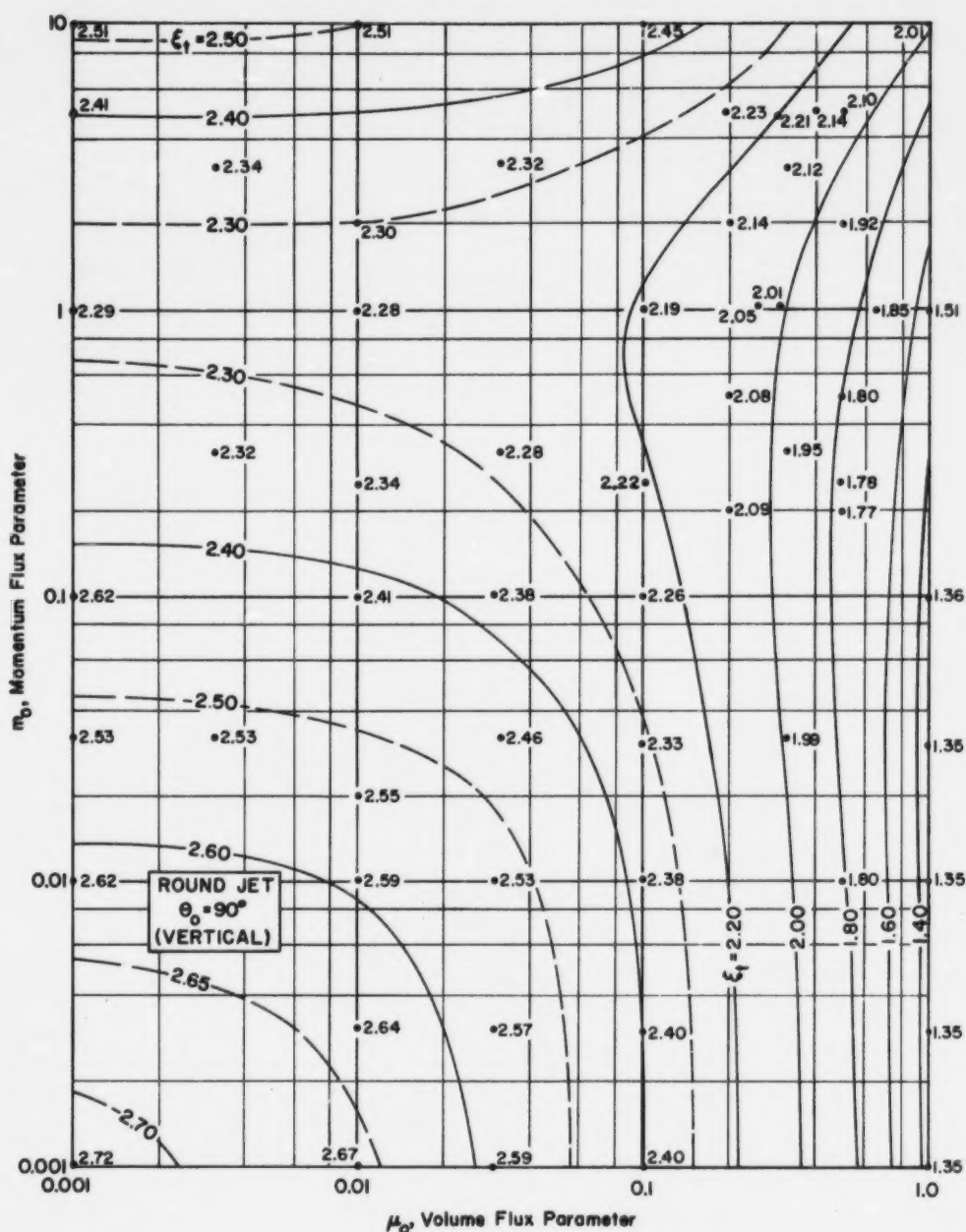


Fig. 12. Terminal Height of Rise ξ_t for Vertical Round Buoyant Jets ($\theta_0 = 90^\circ$) in Linearly Density-Stratified Environments.

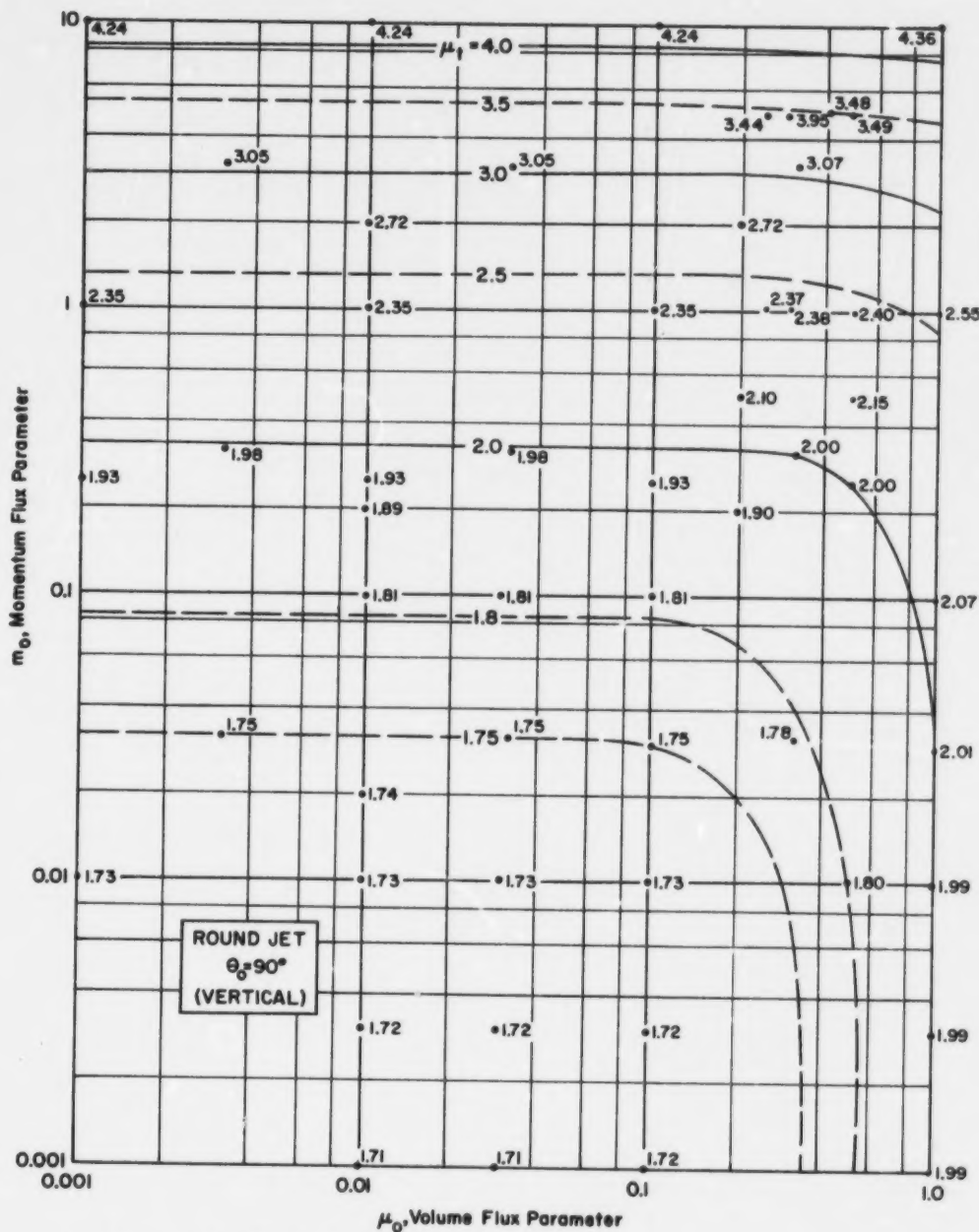


Fig. 13. Terminal Volume Flux Parameter μ_t for Vertical Round Buoyant Jets ($\theta_0 = 90^\circ$) in Linearly Density-Stratified Environments.

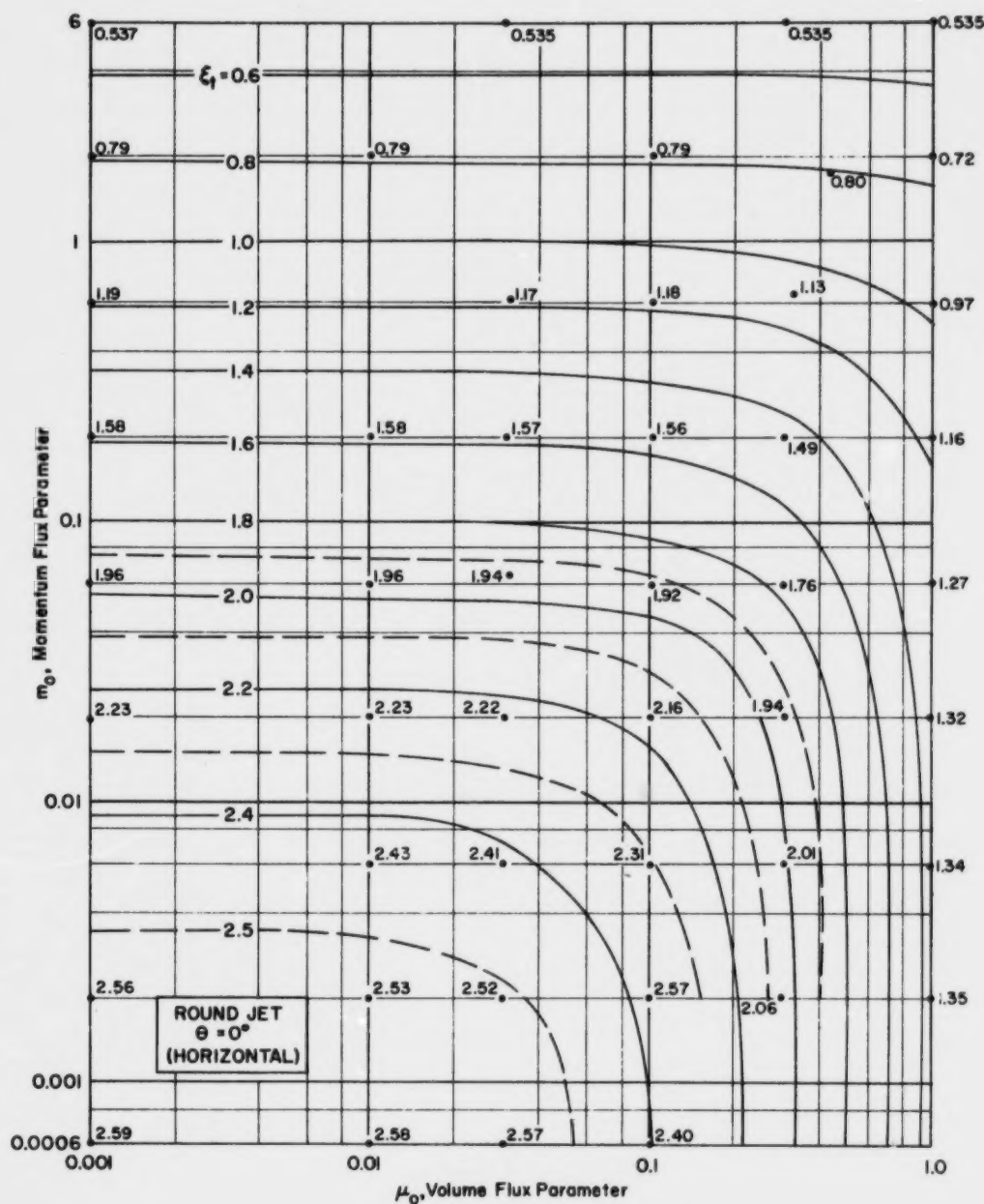


Fig. 14. Terminal Height of Rise ξ_t for Horizontal Round Buoyant Jets ($\theta_o = 0^\circ$) in a Linearly Density-Stratified Environment.

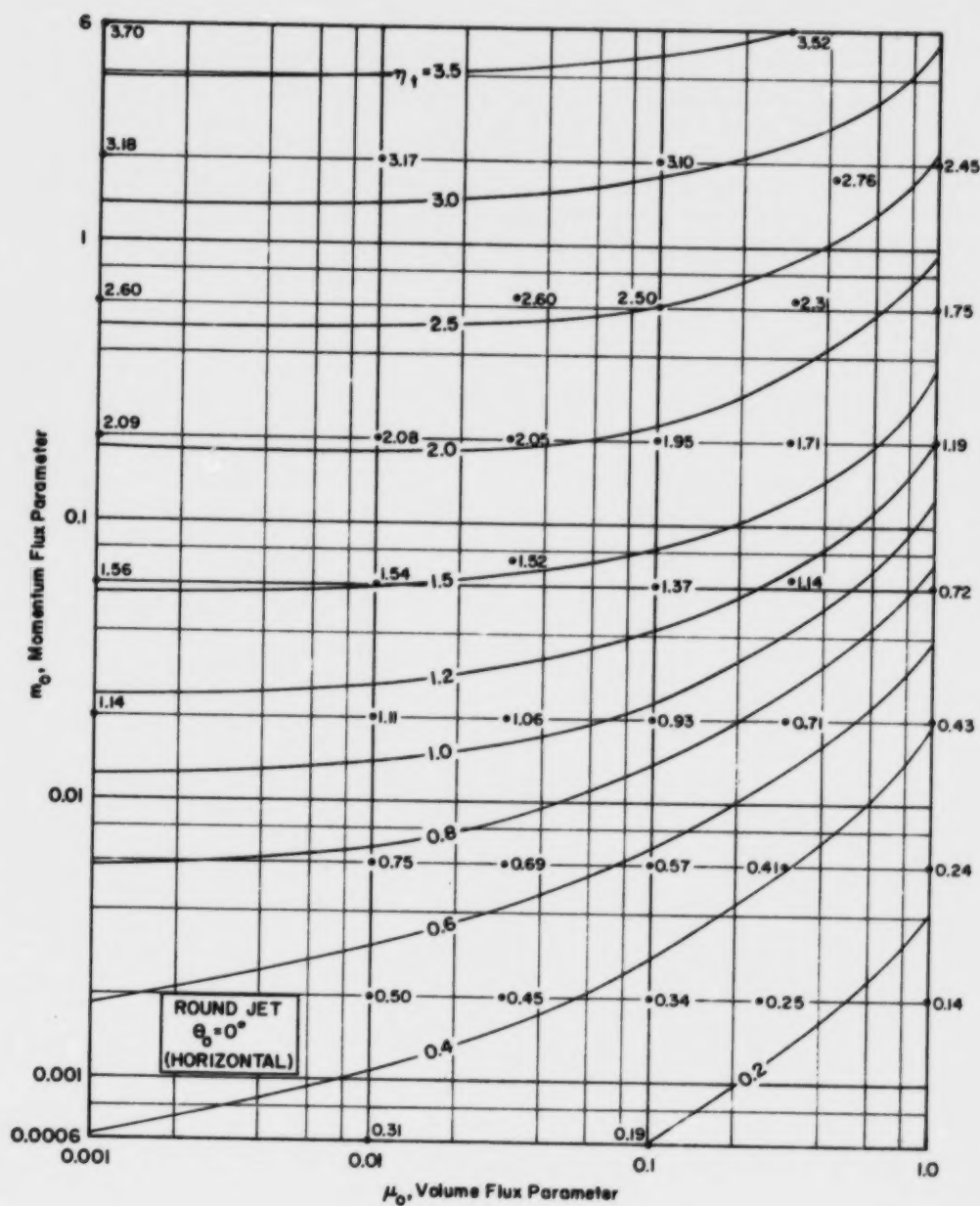
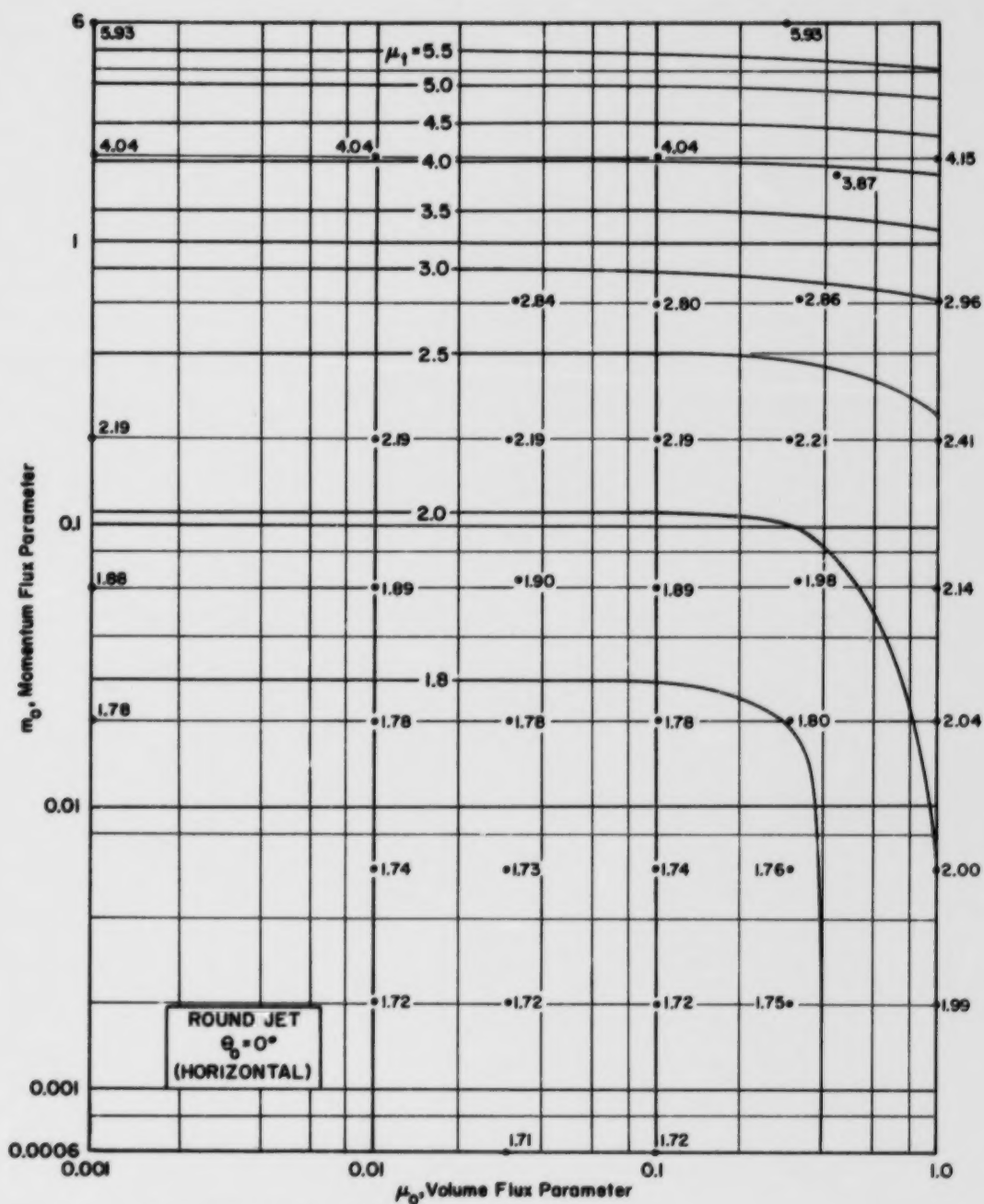


Fig. 15. Terminal Horizontal Coordinate η_t for Horizontal Round Buoyant Jets ($\theta_0 = 0^\circ$) in a Linearly Density-Stratified Environment.



Note: Terminal Dilution Ratio = μ_t / μ_0

Fig. 16. Terminal Volume Flux Parameter μ_t for Horizontal Round Buoyant Jets ($\theta_0 = 0^\circ$).

$$m_o = \frac{(1+\lambda^2)F^2}{4\lambda^4 T} \quad (65)$$

$$\mu_o = \frac{(1+\lambda^2)^{5/8} F^{1/4}}{2\alpha^{1/2} \lambda^{3/2} T^{5/8}} \quad (66)$$

$$F = \frac{2^{1/4} \lambda m_o^{5/8}}{\alpha^{1/2} \mu_o} \quad (67)$$

$$T = \frac{(1+\lambda^2) m_o^{1/4}}{2^{3/2} \alpha \lambda^2 \mu_o^2} \quad (68)$$

The set of parameters m_o and μ_o is adopted in the analysis for mathematical simplicity, and because the solutions found are independent of α and λ values chosen. But in practical applications, it is more convenient to use F and T as parameters.

A scale factor δ defined by

$$\frac{x}{D} = \delta \eta \text{ and } \frac{y}{D} = \delta \xi \quad (69)$$

is related to m_o and μ_o or F and T by:

$$\delta = m_o^{1/4} / (2\sqrt{2} \alpha \mu_o) = \left[\sqrt{\lambda} / (2\sqrt{\alpha} (1+\lambda^2)^{3/8}) \right] F^{1/4} T^{3/8} \quad (70)$$

The coordinate distances from O' are again given by eqs. (47) and (48). The half-width of a buoyant jet may be defined as $w = \sqrt{2} b$, corresponding to two standard deviations from the mean in a Gaussian distribution.

The dimensionless half width of the jet is given by:

$$\begin{aligned} \tilde{w} &= \left\{ G^3 64 \alpha^4 (1+\lambda^2) / F_o^2 \right\} (\sqrt{2} b) \\ &= 2\sqrt{2} \alpha \mu / m_o^{1/4} \text{ for } h_o \neq 0 \end{aligned} \quad (71)$$

At the terminal point, the dilution (eq. (62)) becomes

$$S_t = \frac{2\lambda^2}{1+\lambda^2} \mu_t / \mu_o \quad (72)$$

and in eq. (71) $m = h_o$ at the top, yielding for the half-width:

$$\tilde{w}_t = 2\sqrt{2} \alpha \mu_t / h_o^{1/4} \text{ for } h_o \neq 0 \quad (73)$$

The suggested values for the coefficient of entrainment α and the turbulent spreading ratio λ are respectively 0.082 and 1.16 based upon Rouse et al. 's (23) data for a simple plume. The problem on the coefficient of entrainment for buoyant jets in stratified environments has been discussed in detail by Fan (10). On the basis of limited experimental investigations on jet trajectories and nominal half widths by Fan (10), the choice of $\alpha = 0.082$ and $\lambda = 1.16$ appears reasonable. Future research may indicate some revisions in these values, which can then easily be accommodated into the above results, because α and λ have been carried throughout the analysis explicitly.

With these substitutions of $\alpha = 0.082$ and $\lambda = 1.16$, eqs. (62), (65) to (68), (70) and (71) become:

$$S = 1.15 \mu / \mu_o \quad (62a)$$

$$m_o = 0.324 F^2 / T \quad (65a)$$

$$\mu_o = 2.38 F^{1/4} / T^{5/8} \quad (66a)$$

$$F = 4.82 m_o^{5/8} / \mu_o \quad (67a)$$

$$T = 7.52 m_o^{1/4} / \mu_o^2 \quad (68a)$$

$$\delta = 4.32 m_o^{1/4} / \mu_o = 1.37 F^{1/4} T^{3/8} \quad (70a)$$

$$\tilde{w} = 0.232 \mu / h_o^{1/4} \quad (71a)$$

Also, for the terminal point (maximum height of rise), eq. (72) and (73) become:

$$S_t = 1.15 \mu_t / \mu_o \quad (72a)$$

$$\tilde{w}_t = 0.232 \mu_t / h_o^{\frac{1}{4}} \quad (73a)$$

As an example, consider a horizontal jet of sewage effluent of initial diameter 1.0 ft., discharging at an initial velocity 20 fps into a stratified ocean with a constant density gradient

$$\left(-\frac{1}{\rho_o} \frac{d\rho_a}{dy} \right) = 1.0 \times 10^{-5} / \text{ft.} \quad \text{Then:}$$

$$A_o = 0^\circ, \quad D = 1 \text{ ft}, \quad U_o = 20 \text{ fps}$$

$$T = \frac{\rho_o - \rho_1}{D \left(-\frac{d\rho_a}{dy} \right)} = 2,600 \quad \text{if } \rho_o = 1.026 \text{ gr/c.c.} \\ \text{and } \rho_1 = 1.000 \text{ gr/c.c.}$$

$$F = \frac{U_o}{\sqrt{gD \frac{\rho_o - \rho_1}{\rho_o}}} = 22$$

From eqs. (65a), (66a) and (70a):

$$m_o = 0.068, \quad \mu_o = 0.039 \text{ and } \delta = 56$$

From Figs. 14, 15 and 16

$$\xi_t = 1.94, \quad \eta_t = 1.50 \text{ and } \mu_t = 1.90$$

From eqs. (72a) and (73a)

$$S_t = 1.15 \mu / \mu_o = 55$$

$$\tilde{w}_t = 0.86$$

The location of the terminal point of the jet is then:

$$y'_t = \delta \xi_t D = 109 \text{ ft}$$

$$x'_t = 6.2D + \delta \eta_t D = 93 \text{ ft}$$

with a terminal half width

$$w_t = \delta \tilde{w}_t D = 47 \text{ ft}$$

Thus the jet flow field can be completely submerged if the discharge point is more than $y'_t + w_t = 156 \text{ ft.}$ deep. The initial jet mixing also dilutes the effluent 55 times at the jet centerline.

It is perhaps ambiguous whether or not the maximum upward penetration should be $y'_t + w_t$ (as depicted in Fig. 2b) or simply y'_t . If the jet is nearly vertical at the top (i. e. $\eta_t \ll \xi_t$), the whole flow pattern collapses, and by the theory the width suddenly becomes very large; in such a case it is appropriate to use just y'_t because the central core of the jet penetrates to the highest level, and the peripheral zones fan out before reaching the top. On the other hand, if the jet has appreciable horizontal momentum at the maximum height of rise (i. e. $\eta_t > \xi_t$ or $\eta_t \sim \xi_t$, as in the example above), it will preserve its columnar flow pattern (as in Fig. 2b); in this case, the maximum vertical penetration will be more appropriately taken as $y'_t + w_t$. Until more research is done on the details of the flow at the top of jet or plume, it is not possible to give definitive general recommendations.

CHAPTER V

A TWO-DIMENSIONAL SLOT BUOYANT JET IN A UNIFORM AMBIENT FLUID

In this chapter, theoretical solutions are given for two dimensional inclined buoyant jets in a stagnant, uniform environment. The general approach closely follows the method used in Chapter III except here the jet is issued from a slot of infinite length.

V-A. Formulation of the Problem

1. Conservation equations

The equation of continuity is:

$$\frac{d}{ds} \int_{-\infty}^{\infty} u^* dn = 2\alpha u \quad (74)$$

By substituting u^* from eq. (6), the above equation can be written as:

$$\frac{d}{ds} (ub) = 2\alpha u / \sqrt{\pi} \quad (\text{continuity}) \quad (75)$$

The horizontal momentum flux must be conserved since the pressure is hydrostatic and there is no force acting in the horizontal direction:

$$\frac{d}{ds} \int_{-\infty}^{\infty} \rho^* u^* (u^* \cos \theta) dn = 0 \quad (76)$$

After substituting u^* from eq. (6) and neglecting variations in ρ^* :

$$\frac{d}{ds} \left(\frac{u^2 b}{\sqrt{2}} \cos \theta \right) = 0 \quad (\text{x-momentum}) \quad (77)$$

In the vertical direction, the rate of change of y-momentum flux is equal to the buoyancy force acting on the jet:

$$\frac{d}{ds} \int_{-\infty}^{\infty} \rho^* u^* (u^* \sin \theta) dn = g \int_{-\infty}^{\infty} (\rho_0 - \rho^*) dn \quad (78)$$

Simplifying by substituting eqs. (6) and (7a), eq. (78) then becomes:

$$\frac{d}{ds} \left(\frac{u^2 b}{\sqrt{2}} \cos \theta \right) = g \lambda b \frac{\rho_o - \rho}{\rho_o} \quad (\text{y-momentum}) \quad (79)$$

Conservation of density deficiency can be expressed as:

$$\frac{d}{ds} \int_{-\infty}^{\infty} u^* (\rho_o - \rho^*) dn = \sigma u \left[(\rho_o - \rho_a^* (s, b)) + (\rho_o - \rho_a^* (s, -b)) \right] \quad (80)$$

In the case of uniform ambient density, eq. (80) becomes:

$$\frac{d}{ds} [u b (\rho_o - \rho)] = 0 \quad (\text{density deficiency}) \quad (81)$$

indicating conservation of buoyancy flux.

The relation of continuity for a certain tracer substance present in the source flow only is:

$$\frac{d}{ds} \int_{-\infty}^{\infty} c^* u^* dn = 0 \quad (82)$$

Substituting eqs.(6) and (8), eq. (82) can be written as:

$$\frac{d}{ds} (c u b) = 0 \quad (\text{continuity of tracer}) \quad (83)$$

2. Geometric relations

The jet trajectory is again a part of the solution given by:

$$\frac{dx}{ds} = \cos \theta \quad (\text{geometry}) \quad (84)$$

$$\frac{dy}{ds} = \sin \theta \quad (\text{geometry}) \quad (85)$$

Thus, the problem has seven unknowns, namely, u , b , $\rho_o - \rho$, c , θ , x and y to be solved from seven equations, i.e., (75), (77), (79), (81), (83), (84) and (85).

3. Initial conditions

The initial conditions given at the origin O for the system of ordinary differential equations are:

$$\begin{aligned} u(0) = U_o, \quad b(0) = b_o, \quad \rho(0) = \rho_1, \quad \theta_o(0) = \theta_o \\ c(0) = c_o, \quad x = 0 \text{ and } y = 0 \text{ at } s = 0 \end{aligned} \quad (86)$$

Eqs. (77), (81) and (83) can be integrated immediately. The equation of x-momentum is:

$$\frac{u^2 b}{\sqrt{2}} \cos \theta = \text{const.} \quad (\text{x-momentum}) \quad (87)$$

Eq. (81) for density deficiency flux is then:

$$ub(\rho_o - \rho) = \text{const.} = U_o b_o (\rho_o - \rho_1) \quad (\text{density deficiency}) \quad (88)$$

Eq. (83) for tracer continuity is:

$$cub = \text{const.} = c_o U_o b_o \quad (\text{continuity of a tracer}) \quad (89)$$

Thus $(\rho_o - \rho)$ and c are simply related to the solutions for u and b as given by eqs. (88) and (89). These two equations need not be carried through the following calculation.

V-B. Normalized Equations and Dimensionless Parameters

In order to transform the system of equations into a simple, normalized form, dimensionless parameters are defined as follows:

Volume flux parameter: $\mu = ub/(U_o b_o) \quad (90)$

Momentum flux parameters:

$$m = \left\{ \frac{4\alpha \rho_0}{\sqrt{\pi} \lambda g U_0^4 b_0^4 (\rho_0 - \rho_1)} \right\}^{\frac{1}{3}} \frac{u^2 b}{\sqrt{2}} \quad (91)$$

$$h = m \cos \theta \quad (92)$$

$$v = m \sin \theta \quad (93)$$

Coordinates:

$$s: \quad \zeta = \left\{ \frac{4\sqrt{2} g \alpha^2 \lambda (\rho_0 - \rho_1)}{\pi \rho_0 U_0^2 b_0^2} \right\}^{\frac{1}{3}} s \quad (94)$$

$$x: \quad \eta = \left\{ \frac{4\sqrt{2} g \alpha^2 \lambda (\rho_0 - \rho_1)}{\pi \rho_0 U_0^2 b_0^2} \right\}^{\frac{1}{3}} x \quad (95)$$

$$y: \quad \xi = \left\{ \frac{4\sqrt{2} g \alpha^2 \lambda (\rho_0 - \rho_1)}{\pi \rho_0 U_0^2 b_0^2} \right\}^{\frac{1}{3}} y \quad (96)$$

By substitution, eqs. (75), (77), (79), (84) and (85) then become:

$$\frac{d\mu}{d\zeta} = m/\mu \quad (97)$$

$$h = \sqrt{m^2 - v^2} = \text{const.} = h_0 \quad (98)$$

$$\frac{dv}{d\zeta} = \mu/m \quad (99)$$

$$\frac{d\eta}{d\zeta} = h/m \quad (100)$$

$$\frac{d\xi}{d\zeta} = v/m \quad (101)$$

The corresponding initial conditions (86) are:

$$\begin{aligned} \mu(0) &= 1, \quad m(0) = m_0, \quad \theta(0) = \theta_0, \\ \eta &= 0 \text{ and } \xi = 0 \text{ at } \zeta = 0 \end{aligned} \quad (102)$$

This set of equations was again solved numerically using the same method as described in Section III-B. In this problem, there are two relevant parameters θ_0 and m_0 . For each combination of these two parameters, a computer run was made until the value of ξm_0 reached 50. A wide range of θ_0 and m_0 values was covered in this study. Results are presented in the next section.

V-C. Solutions of Jet Gross Behavior

Numerical solutions were carried out for cases of various angles of discharge: $\theta_0 = 0^\circ$ (horizontal discharge), 15° , 30° , 45° , 60° , and 90° (vertical discharge) over a range of $m_0 = 0.56$ to 14.1 . Figs. 17 (a) to (d) are graphs of jet trajectories (ξm_0 vs. ηm_0) for cases $\theta_0 = 0^\circ$, 15° , 30° and 45° respectively. Normalized jet half-width b/b_0 , are also shown in these graphs. For $\theta_0 = 90^\circ$, only b/b_0 values are plotted as shown in Fig. 17(e).

The coordinates ξm_0 and ηm_0 are convenient because of the following relationships:

$$\frac{x}{b_0} = \frac{\sqrt{\pi} \eta}{2\alpha} m_0; \quad \frac{y}{b_0} = \frac{\sqrt{\pi} \xi}{2\alpha} m_0 \quad (103a)$$

The effects of various parameters on jet trajectories are similar to the case of round jet.

The centerline dilution ratio S_0 , i. e. along jet axis, with respect to the centerline concentration at O, from eqs. (89) and (90) is:

$$S_0 = c_0/c = \mu \quad (103b)$$

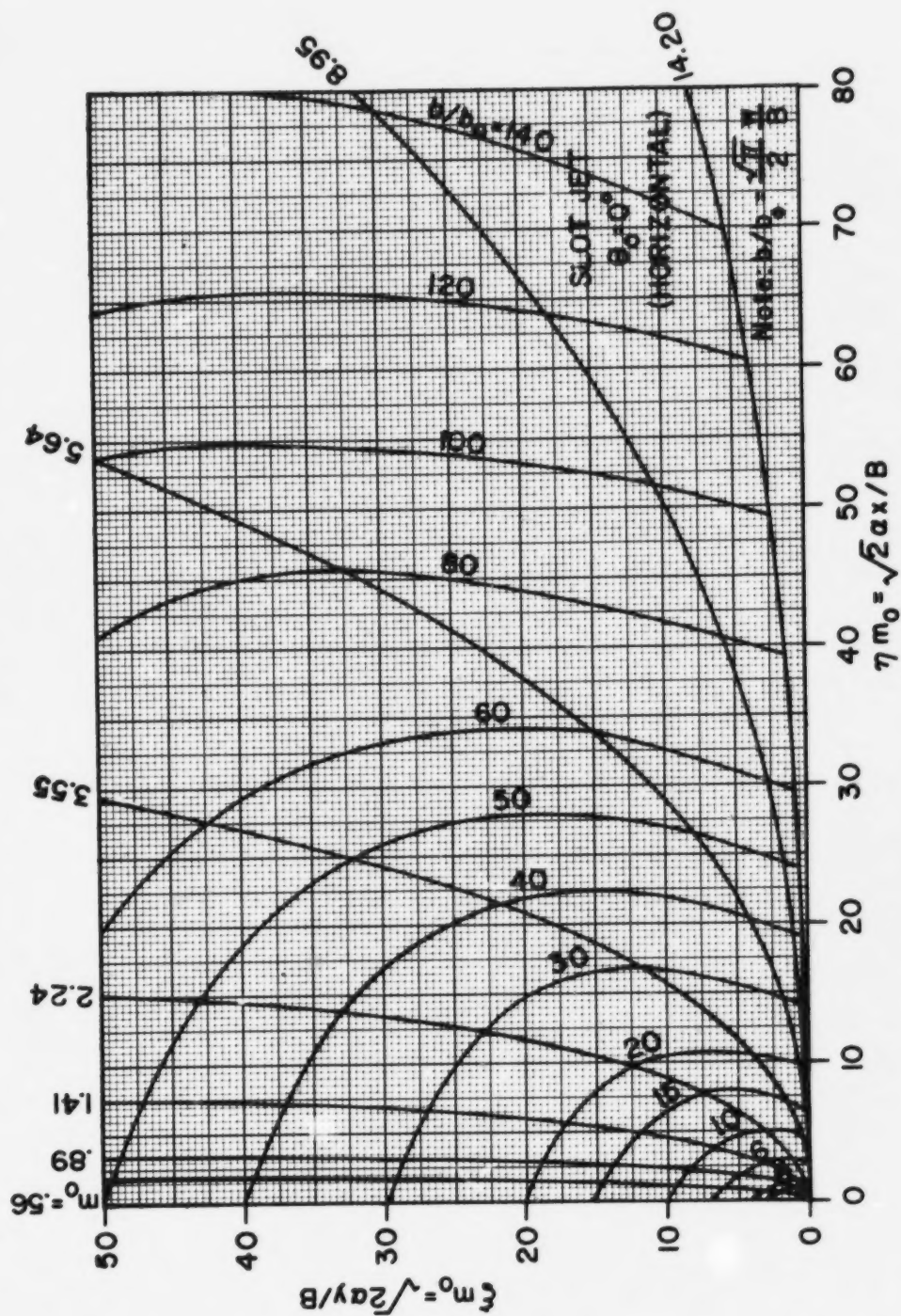


Fig. 17(a). Trajectories and Half-Widths, b/b_0 , for Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 0^\circ$.

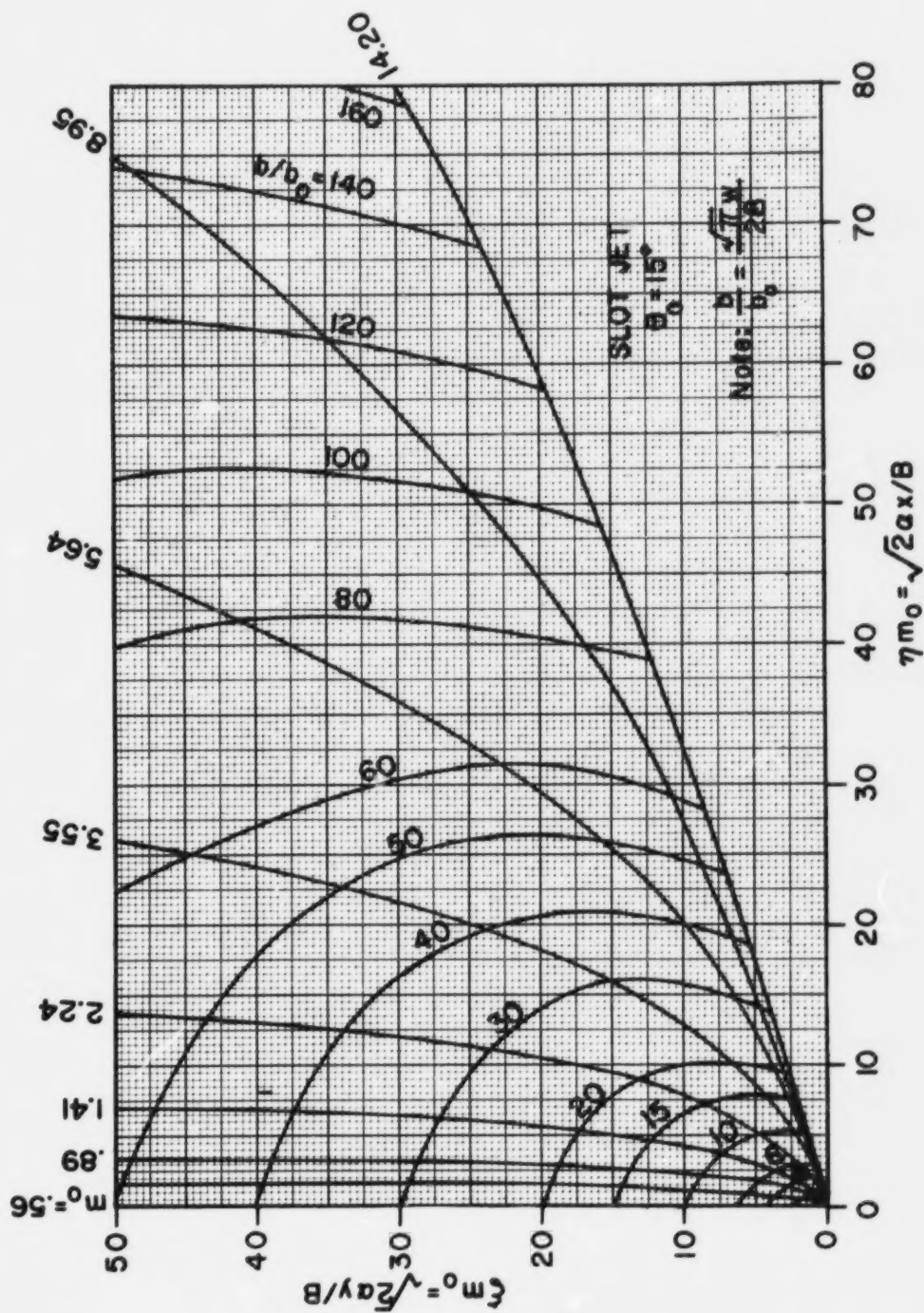


Fig. 17(b). Trajectories and Half-Widths, b/b_0 , for Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 15^\circ$.

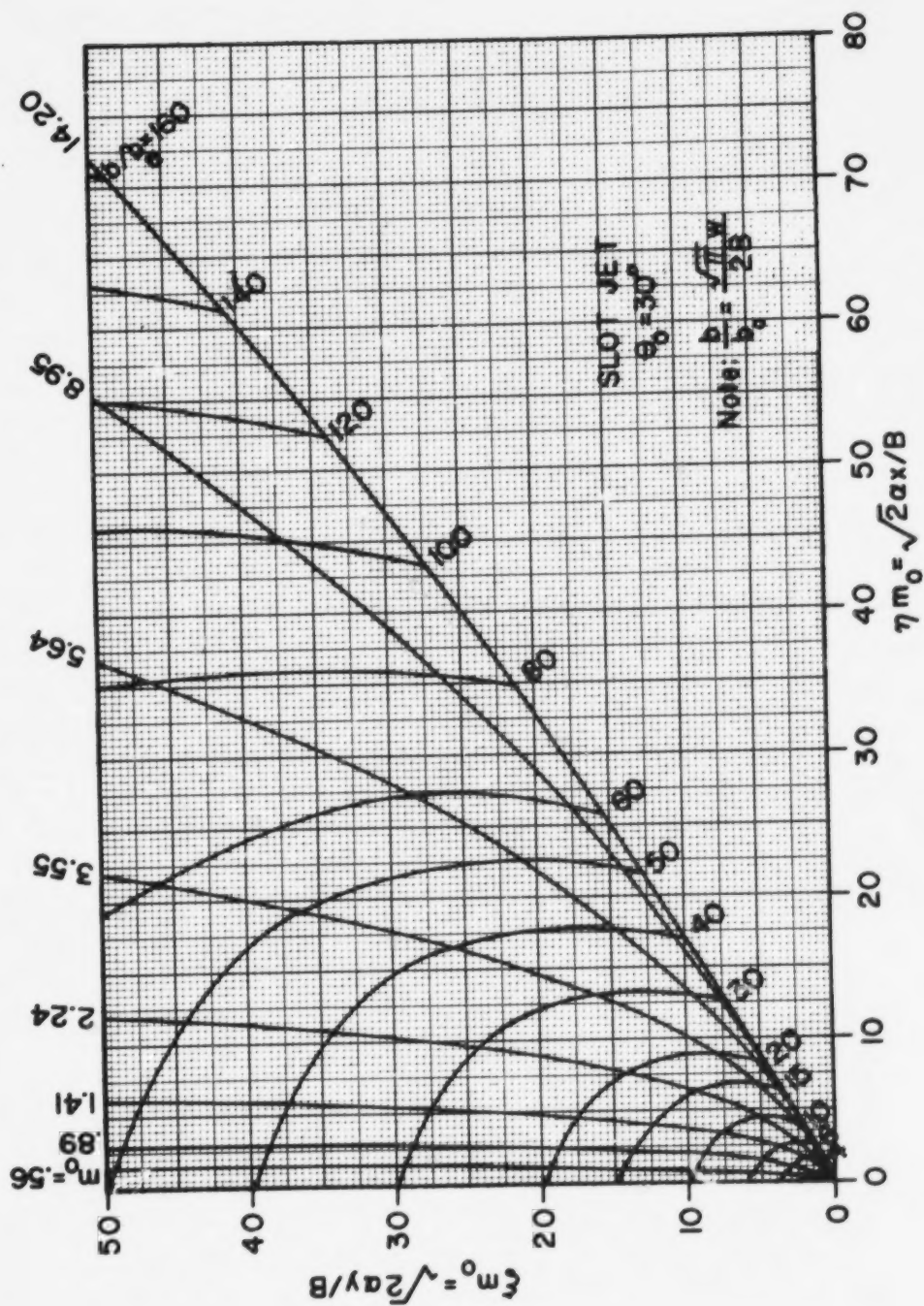


Fig. 17(c). Trajectories and Half-Widths, b/b_0 , for Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 30^\circ$.

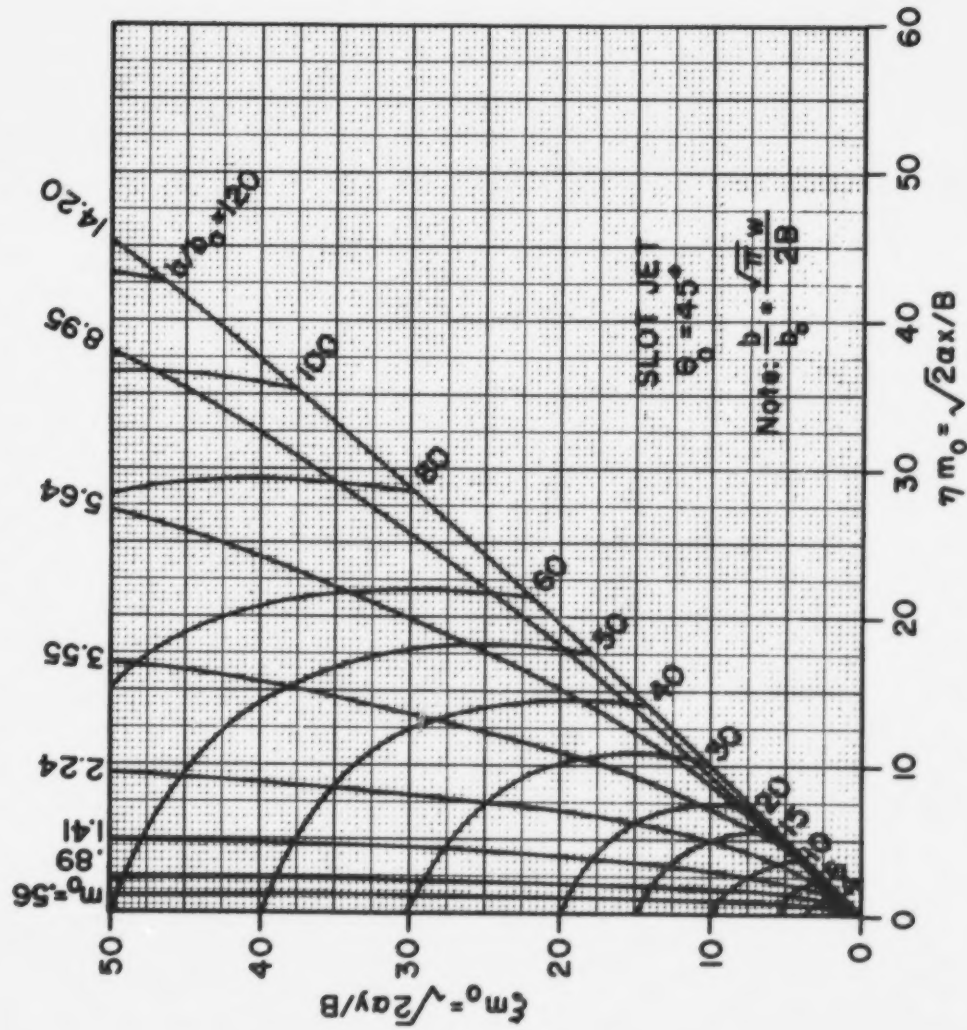


Fig. 17(d). Trajectories and Half-Widths, b/b_0 , for Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 45^\circ$.

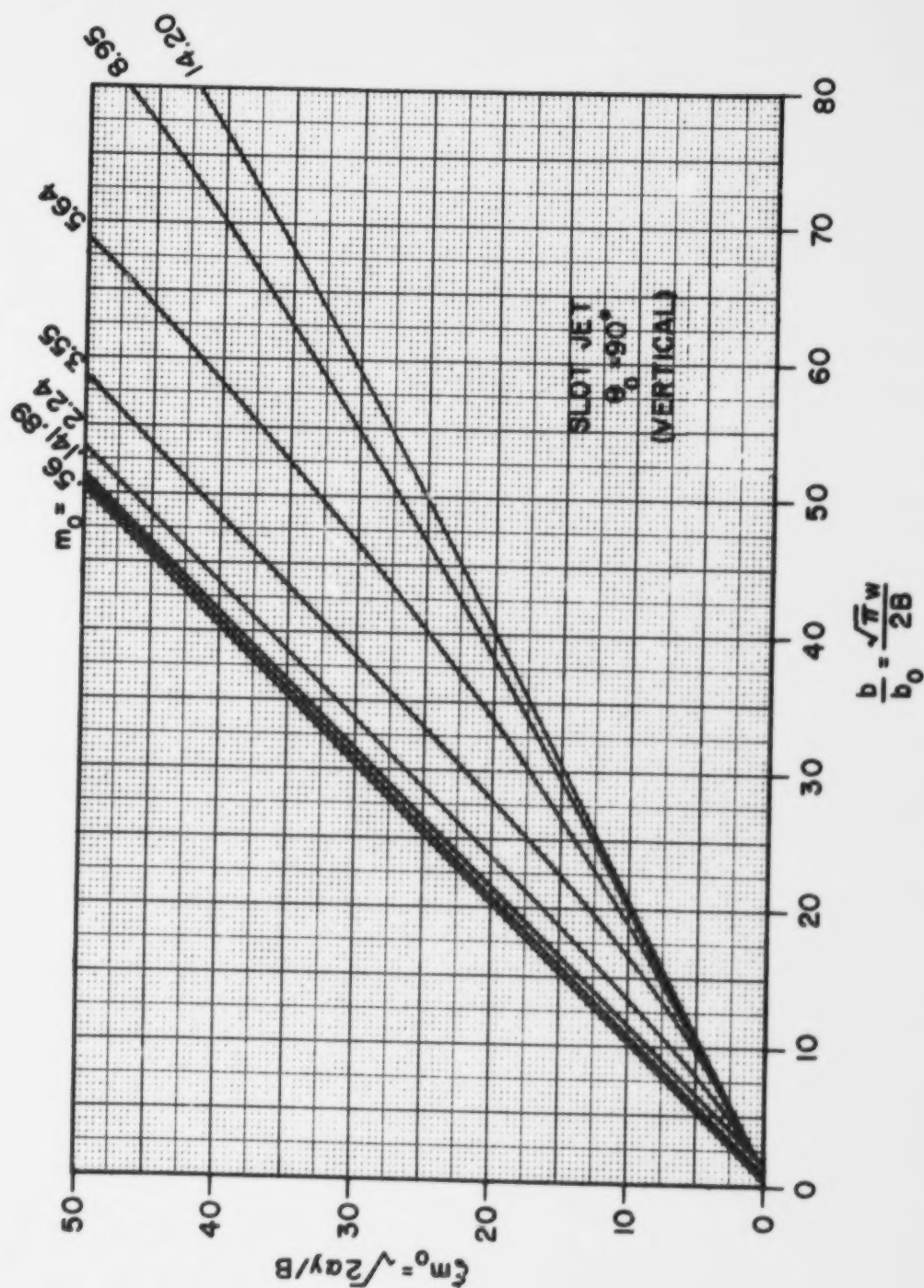


Fig. 17(e). Half-Widths, b/b_0 , for Slot Buoyant Jets in Stagnant Uniform Environments:
 $\theta_0 = 90^\circ$. (Trajectories are all vertical lines.)

Figs. 18 (a) to (f) are dilution graphs showing S_o as functions of m_o and ξm_o for different θ_o cases. As in the round jet case, there are regions of minimum dilution at some ξm_o and m_o values for non-vertical jets. Dilution values at ξm_o greater than 50 can be obtained by utilizing the simple plane plume relation: $S_o \approx \xi$ in small and intermediate m_o cases.

Figs. 17 and 18 represent gross behavior of slot buoyant jets. Practical application of these solutions are discussed in the following section.

V-D. Application to Practical Problems

To apply the results it is necessary to consider the zone of flow establishment. Again Albertson et al.'s (5) results on an ordinary slot jet are adopted for a buoyant slot jet with buoyancy neglected. The length of the zone is taken to be $5.2 B$ where B is the initial jet width as shown in Fig. 19. The initial top-hat profiles at O' develop into Gaussian distributions given by eqs. (6), (7) and (8) at the end of the zone.

The initial value b_o can be obtained by applying the momentum relation between cross sections O and O' assuming that the buoyancy force is negligible in such a short region:

$$\begin{aligned} U_o^2 B &= \int_{-\infty}^{\infty} u^2 dn \\ &= \sqrt{\frac{\pi}{2}} U_o^2 b_o \end{aligned} \quad (104)$$

Thus:

$$b_o = \sqrt{\frac{2}{\pi}} B \quad (105)$$

The tracer concentration c_o at O , assuming that the concentration at O' is unity, can be obtained by considering the continuity relation for the tracer:

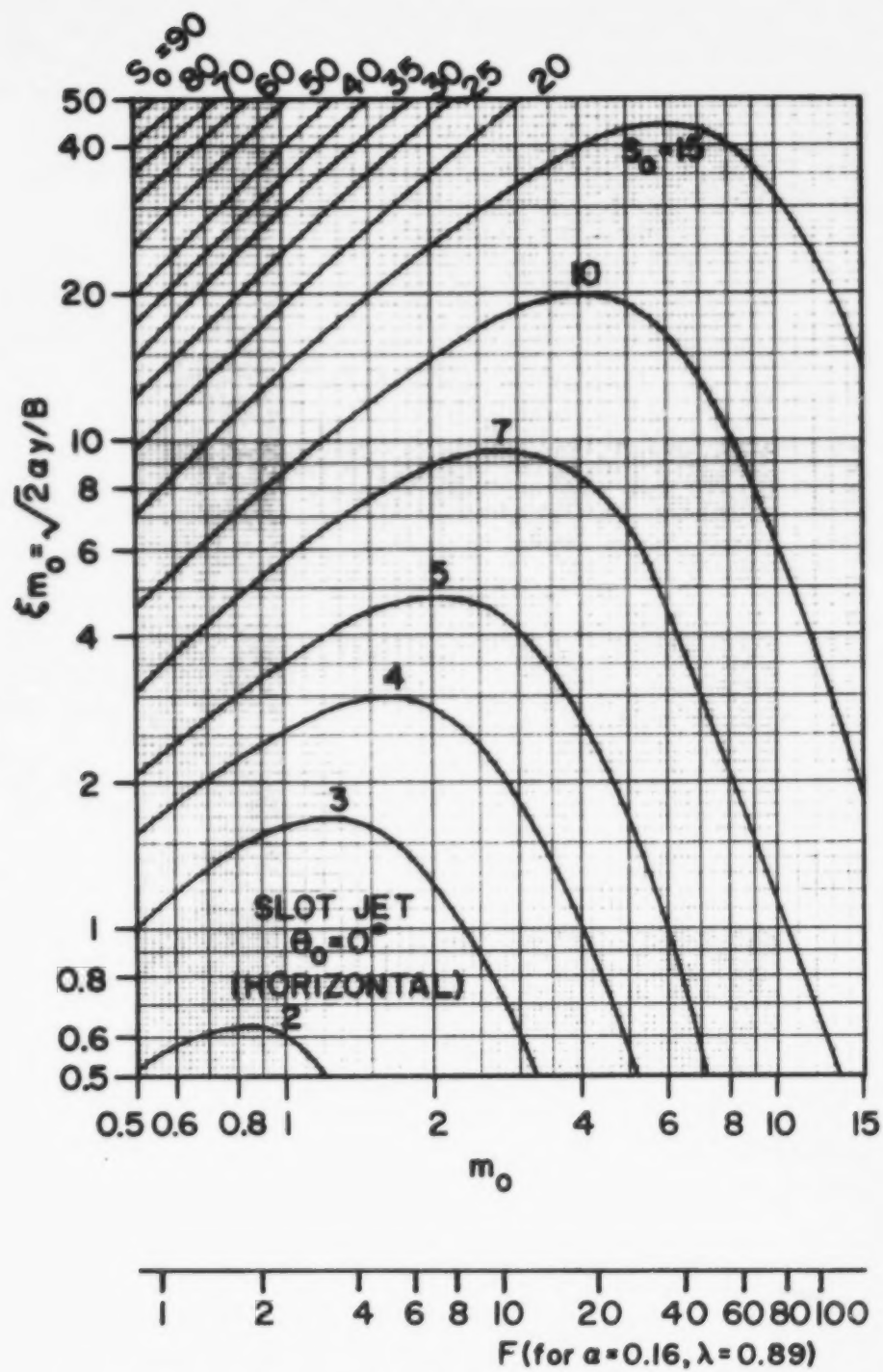


Fig. 18(a). Dilution of Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 0^\circ$.

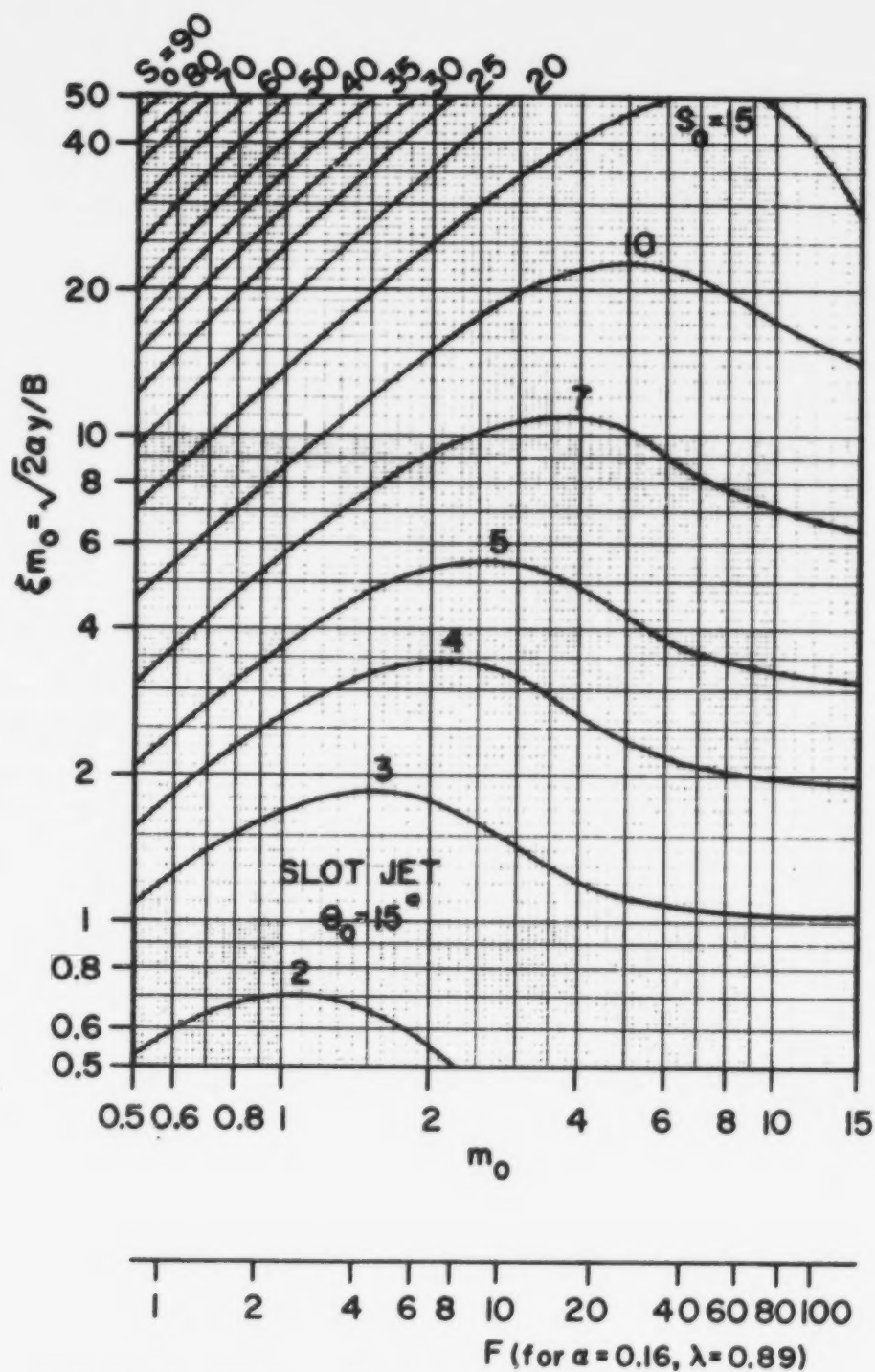


Fig. 18(b). Dilution of Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 15^\circ$.

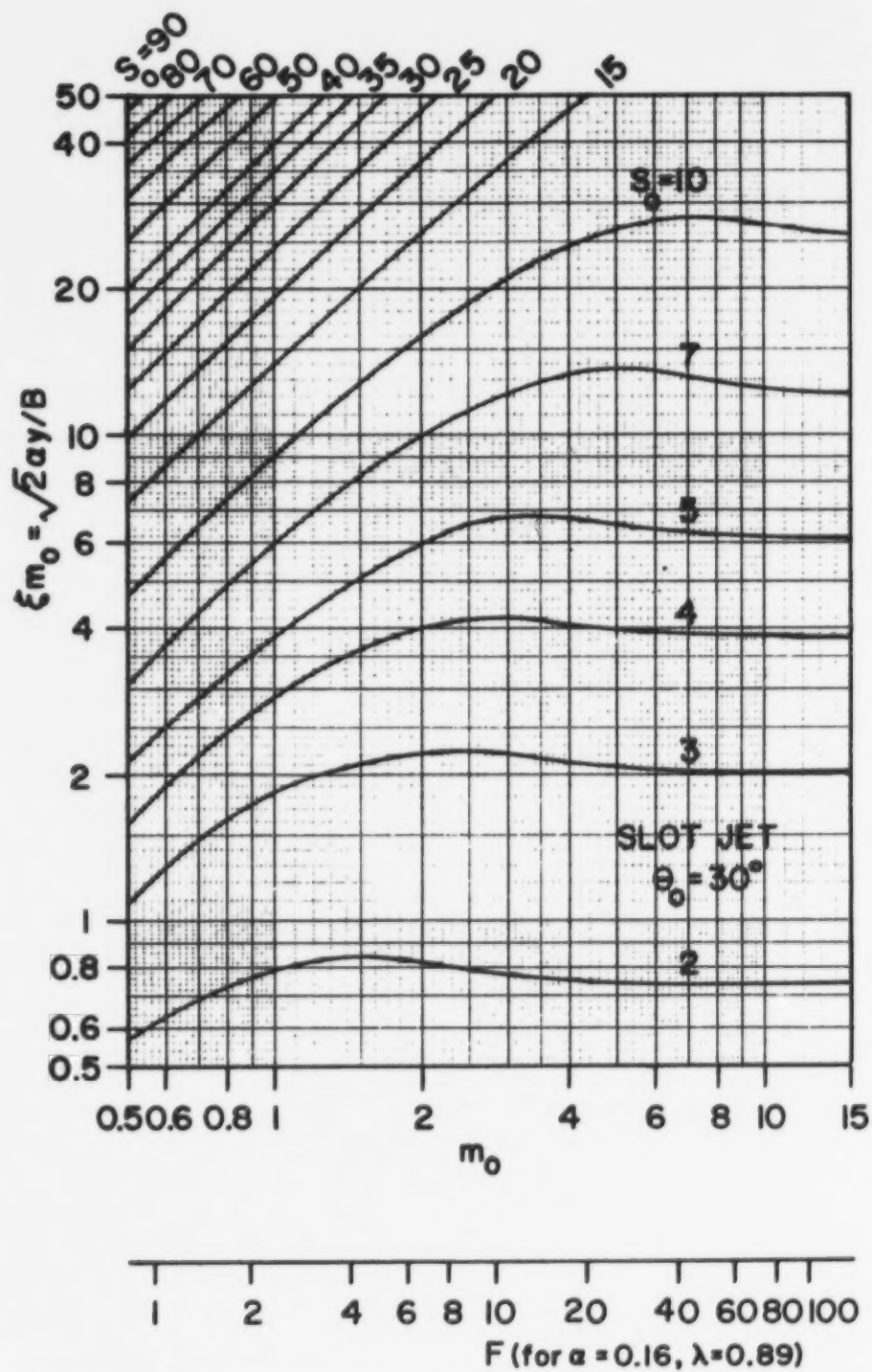


Fig. 18(c). Dilution of Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 30^\circ$.

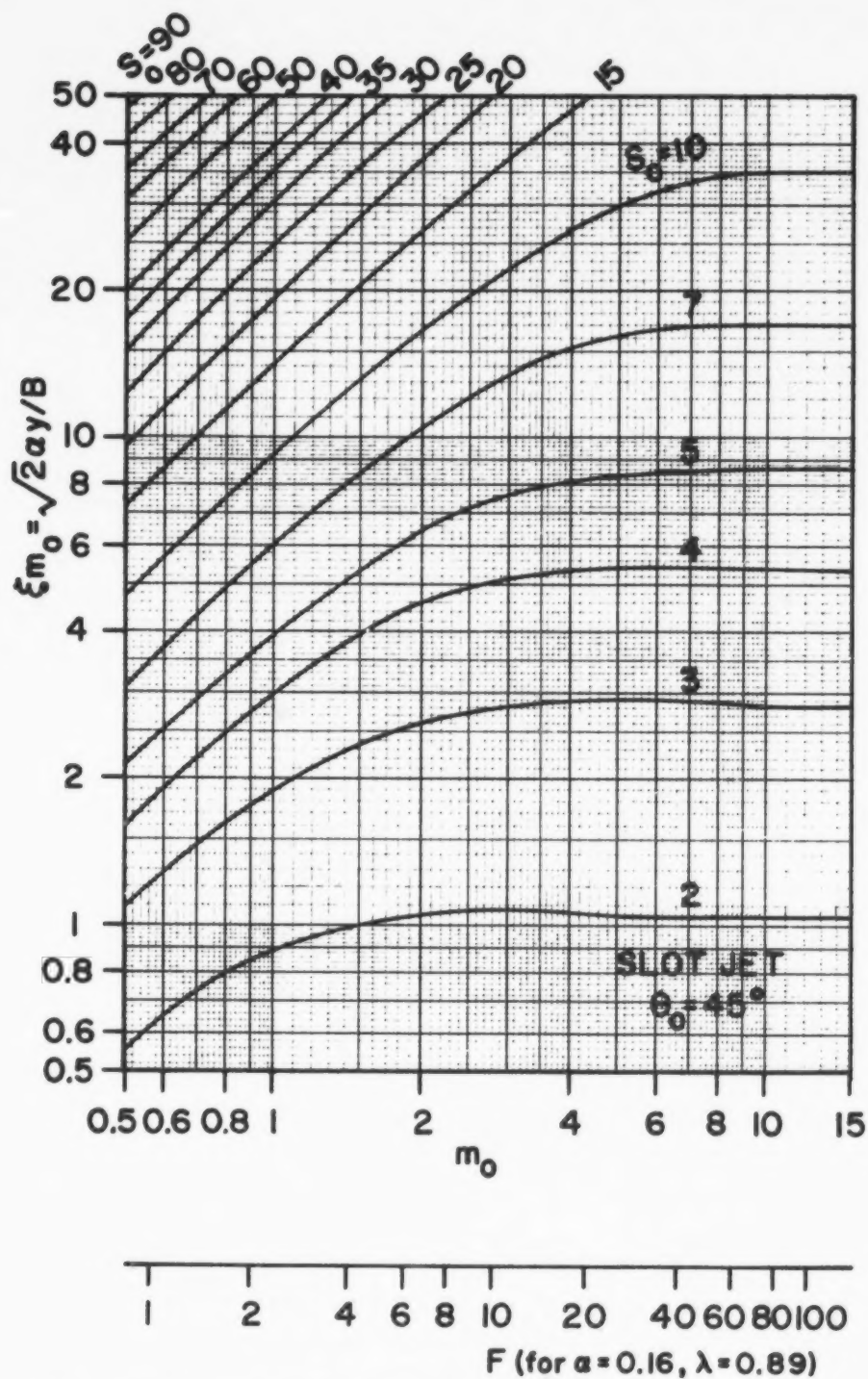


Fig. 18(d). Dilution of Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 45^\circ$.

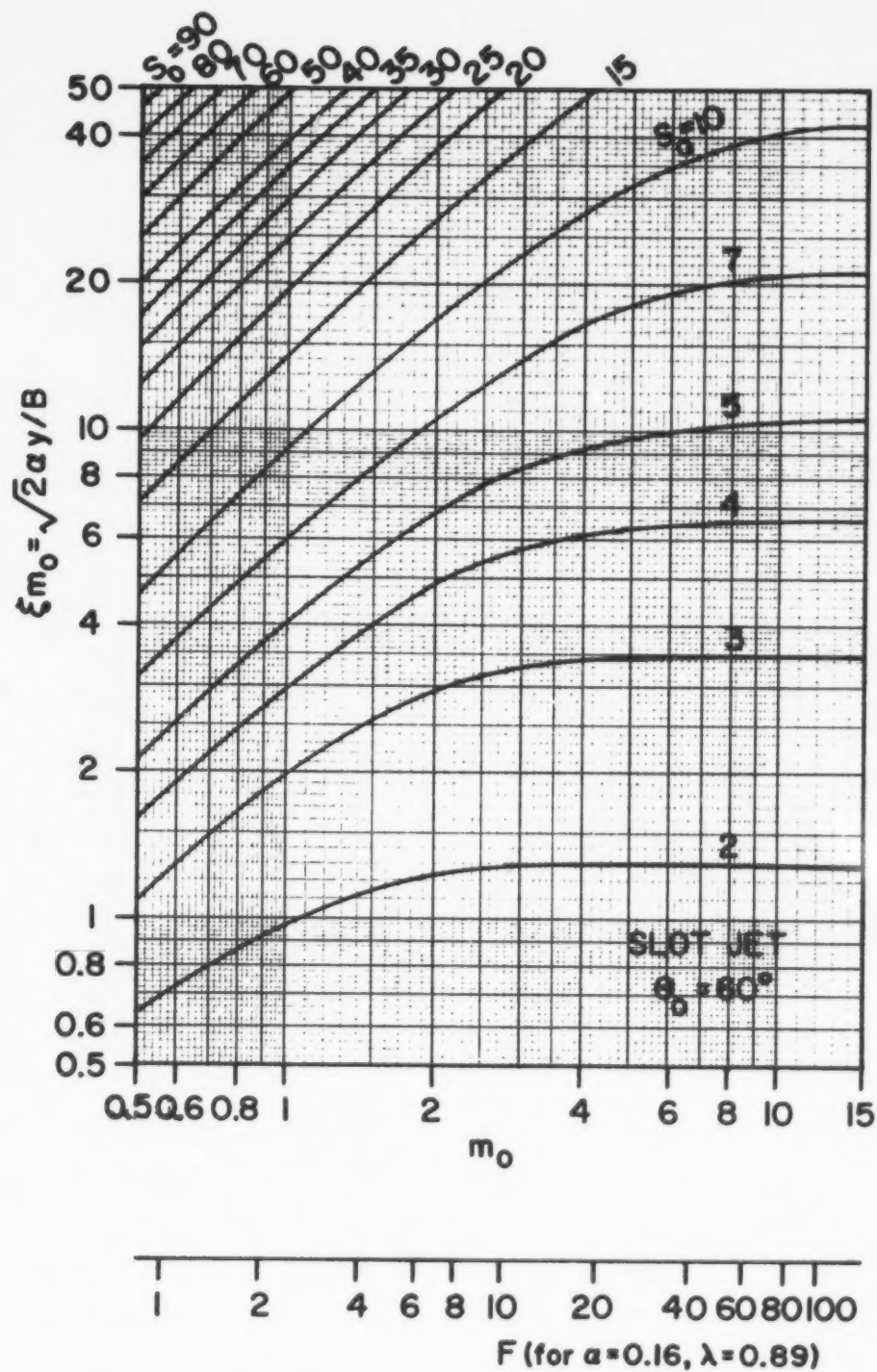


Fig. 18(e). Dilution of Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 60^\circ$.

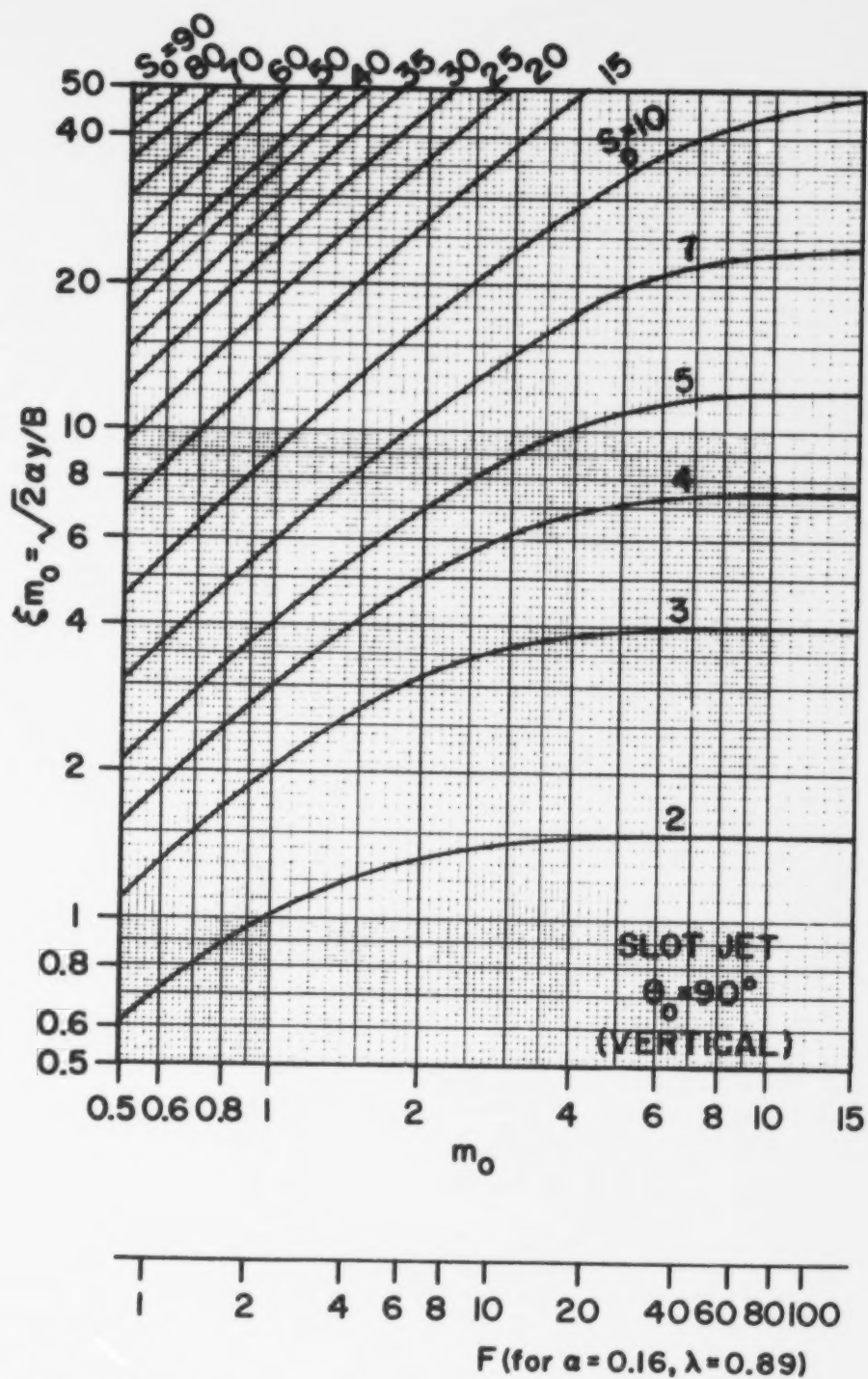


Fig. 18(f). Dilution of Slot Buoyant Jets in Stagnant Uniform Environments: $\theta_0 = 90^\circ$.

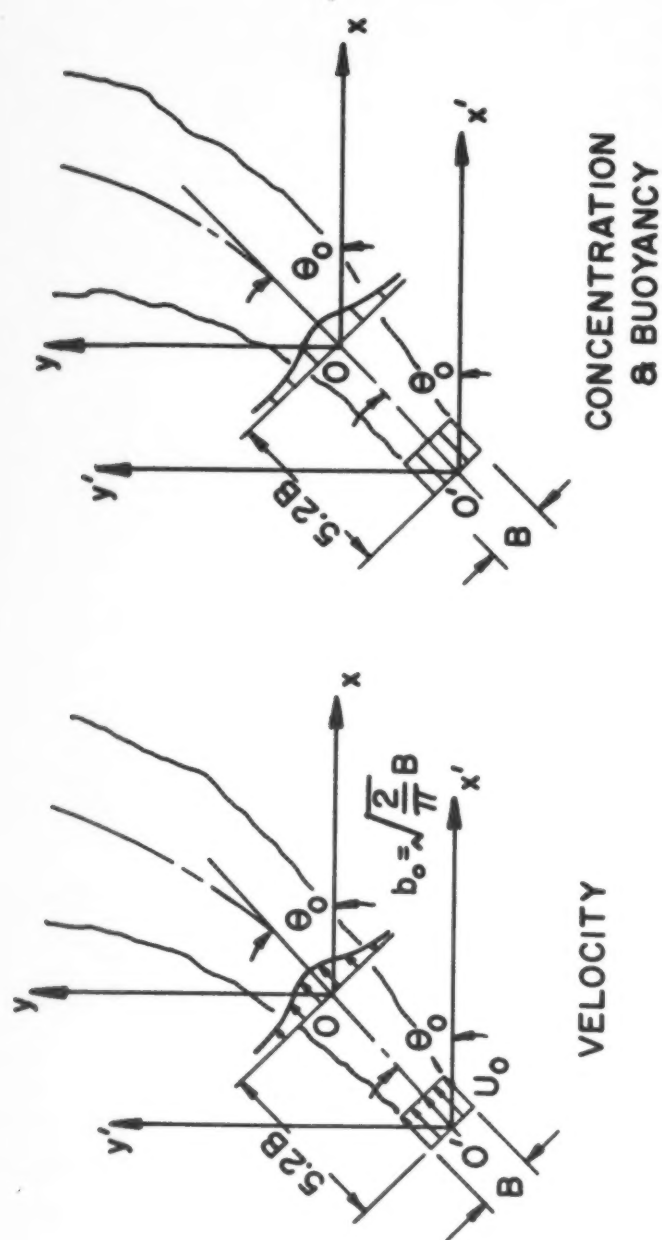


Fig. 19. Zone of Flow Establishment for an Inclined Slot Buoyant Jet.

$$U_o B \cdot l = \int_{-\infty}^{\infty} u \cdot c \cdot dn$$

$$= \sqrt{\frac{\lambda^2 \pi}{1+\lambda^2}} c_o U_o b_o$$
(106)

thus,

$$c_o = \sqrt{(1+\lambda^2)/(2\lambda^2)}$$
(107)

If a densimetric jet Froude number is defined as:

$$F = \frac{U_o}{\sqrt{\frac{\rho_o - \rho_1}{\rho_o} g B}}$$
(108)

it can be shown that F is directly related to m_o by:

$$F = \sqrt{\frac{\lambda}{\alpha}} m_o^{3/2}, \text{ or } m_o = \left(\frac{\alpha}{\lambda} F^2\right)^{1/3}$$
(109)

Thus the problem of an inclined slot buoyant jet in a stagnant uniform environment is governed by two parameters, namely, either F and θ_o or m_o and α_o . The dimensionless distances x/B and y/B are related to the quantities ηm_o and ξm_o by:

$$x/B = \frac{1}{\sqrt{2} \alpha} \eta m_o$$
(110)

$$y/B = \frac{1}{\sqrt{2} \alpha} \xi m_o$$
(111)

The dimensionless distances with respect to the discharge point O' , i.e. x'/B and y'/B are given by

$$x'/B = x/B + 5.2 \cos \theta_o$$
(112)

$$y'/B = y/B + 5.2 \sin \theta_o$$
(113)

Suggested values for α and λ for two-dimensional buoyant jets are:

$$\alpha = 0.16 \text{ and } \lambda = 0.89$$

based upon results on two-dimensional buoyant plume experiments by Rouse et al. (23). Since for this choice $\lambda = 0.89$ is less than 1, eq. (107) gives $c_o = 1.06 > 1$. Obviously, it is impossible physically for c_o to be greater than unity. The implication of $\lambda < 1$ is that at the end of zone of flow establishment defined by velocity profile, the concentration profile is not yet fully developed. Because of this, we obtain the dilution ratio S with respect to the discharge point O'

$$S = \frac{1}{c} = \sqrt{\frac{2\lambda^2}{1+\lambda^2}} \frac{c_o}{c} = \sqrt{\frac{2\lambda^2}{1+\lambda^2}} \mu, \text{ if } \sqrt{\frac{2\lambda^2}{1+\lambda^2}} \mu \geq 1 \quad (114)$$

$$\text{or } S = 1, \text{ if } \sqrt{\frac{2\lambda^2}{1+\lambda^2}} \mu < 1$$

By substitution of these suggested values ($\alpha = 0.16$ and $\lambda = 0.89$), eqs. (109) to (111) and (114) become:

$$F = 2.36 m_o^{3/2} \text{ or } m_o = 0.562 F^{2/3} \quad (109a)$$

$$x/B = 4.41 \eta m_o \quad (110a)$$

$$y/B = 4.41 \xi m_o \quad (111a)$$

$$\left. \begin{array}{l} S = 0.94 \mu \text{ for } \mu \geq 1.06 \\ S = 1 \quad \text{for } \mu < 1.06 \end{array} \right\} \quad (114a)$$

In case of a row of closely spaced circular ports, issuing jets of diameter D at spacing L on centers, an equivalent slot width B_e may be defined as

$$B_e = \frac{\pi D^2}{4L}$$

For widely spaced circular ports, the solution must first be obtained as individual round buoyant jets, until they start interfering with each other. At this point an equivalent line source may be approximated.

CHAPTER VI

A TWO-DIMENSIONAL SLOT BUOYANT JET IN A LINEARLY DENSITY-STRATIFIED AMBIENT FLUID

In this chapter, theoretical solutions are obtained for a slot buoyant jet in a stagnant environment with linear density gradient. The analysis has been previously given by Brooks and Koh (8) for vertical slot jets, and by Fan (11) for inclined slot jets. More numerical results are incorporated here.

VI-A. Formulation of the Problem

1. Conservation equations

The basic conservation equations are the same as eqs. (74), (76), (80) and (82). For a slot jet in a stagnant environment with linear density gradient eq. (74) reduces to eq. (75), eq. (76) to eq. (77) and eq. (82) to (83) without change. However, in a linearly density-stratified environment, ρ_0 becomes ρ_a in eq. (78), and eq. (79) is revised slightly to:

$$\frac{d}{ds} \left(\frac{u^2 b}{\sqrt{2}} \sin \theta \right) = g \lambda b (\rho_a - \rho) / \rho_0 \quad (\text{y-momentum}) \quad (115)$$

and eq. (80) for flux of density deficiency reduces to:

$$\frac{d}{ds} [ub(\rho_a - \rho)] = \sqrt{\frac{1+\lambda^2}{\lambda^2}} ub \frac{d\rho_a}{ds} \quad (\text{density deficiency}) \quad (116)$$

2. Geometric relations

The jet trajectory is again expressed by eqs. (84) and (85). Thus the problem has seven unknown functions of s , namely, u , b , $\rho_a - \rho$, θ , c , x and y to be solved from seven equations, i. e., eqs. (75), (77), (115), (116), (83), (84) and (85).

3. Initial conditions

The initial conditions are unchanged and given by eq. (86). Again the equations of x-momentum and tracer concentration can be integrated as eqs. (87) and (89). Eq. (89) will not be carried through in the integration process.

VI-B. Normalized Equations and Dimensionless Parameters

In order to transform the system of equations into simple normalized form, dimensionless parameters are defined as follows:

Volume flux parameter:

$$u = \left\{ \frac{32 \sqrt{2(1+\lambda^2)} F_o^4 \alpha^2}{\pi G^3} \right\}^{-1/3} (u^2 b^2) \quad (117)$$

Momentum flux parameter:

$$\begin{aligned} m &= \left\{ G / \left[\sqrt{2(1+\lambda^2)} F_o^2 \right] \right\} \left(\frac{u^4 b^2}{2} \right) && \text{in s-direction} \\ h &= m \cos^2 \theta && \text{in x-direction} \\ v &= m \sin^2 \theta && \text{in y-direction} \end{aligned} \quad (118)$$

Buoyancy flux parameter:

$$B = \left\{ \frac{\lambda^2}{1+\lambda^2} / F_o \right\} g u b (\rho_a - \rho) / \rho_o \quad (119)$$

Coordinates:

$$\begin{aligned} s: \quad \zeta &= \left\{ \frac{32 \sqrt{2(1+\lambda^2)} G^3 \alpha^2}{\pi F_o^2} \right\}^{1/6} s \\ x: \quad \eta &= \left\{ \frac{32 \sqrt{2(1+\lambda^2)} G^3 \alpha^2}{\pi F_o^2} \right\}^{1/6} x \\ y: \quad \xi &= \left\{ \frac{32 \sqrt{2(1+\lambda^2)} G^3 \alpha^2}{\pi F_o^2} \right\}^{1/6} y \end{aligned} \quad (120)$$

where $G = -\frac{g}{\rho_o} \frac{d\rho_a}{dy}$ and $F_o = \sqrt{\frac{\lambda^2}{1+\lambda^2}} g U_o b_o \frac{\rho_o - \rho_1}{\rho_o}$ are not dimensionless.

Eqs. (75), (77), (115), (116), (83) to (85) then become:

$$\frac{d\mu}{d\zeta} = \sqrt{m} \quad (121)$$

$$h = m - v = \text{const.} = h_o \quad (122)$$

$$\frac{dv}{d\zeta} = \theta \left(\frac{\mu v}{m}\right)^{\frac{1}{2}} \quad (123)$$

$$\frac{d\theta}{d\zeta} = -\left(\frac{\mu v}{m}\right)^{\frac{1}{2}} \quad (124)$$

$$\frac{d\eta}{d\zeta} = (h/m)^{\frac{1}{2}} \quad (125)$$

$$\frac{d\xi}{d\zeta} = (v/m)^{\frac{1}{2}} \quad (126)$$

The corresponding initial conditions (86) are:

$$\mu(0) = \mu_o, \quad m(0) = m_o, \quad \theta = \theta_o, \quad \theta = 1 \quad (127)$$

$$\eta = 0 \text{ and } \xi = 0 \text{ at } \zeta = 0$$

Numerical solutions were obtained by a technique similar to that described in previous chapters. In this specific problem, the governing parameters are θ_o , μ_o and m_o . For each combination of these parameters, a computer run was made. A wide range of initial conditions was covered by the numerical computation. These results are presented in the next section.

VI-C. Solution of Jet Gross Behavior

Figs. 20 and 21 are jet trajectories for slot buoyant jets with $\mu_o = 0$ at various initial angles of discharge with $m_o = 0.2$ and 2.0 respectively. Terminal heights of rise ξ_t increase with increasing θ_o .

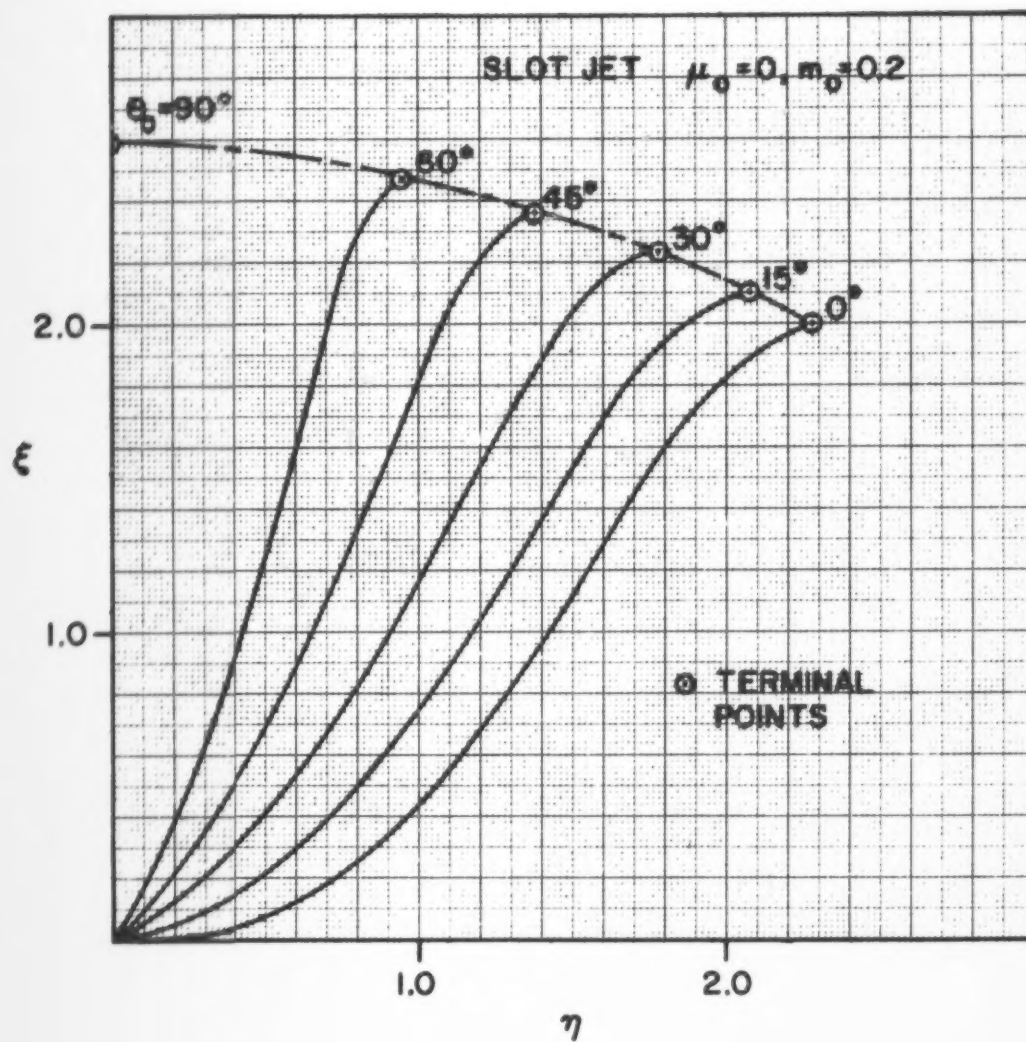


Fig. 20. Paths of Inclined Slot Buoyant Jets with $\mu_0 = 0, m_0 = 0.2$.

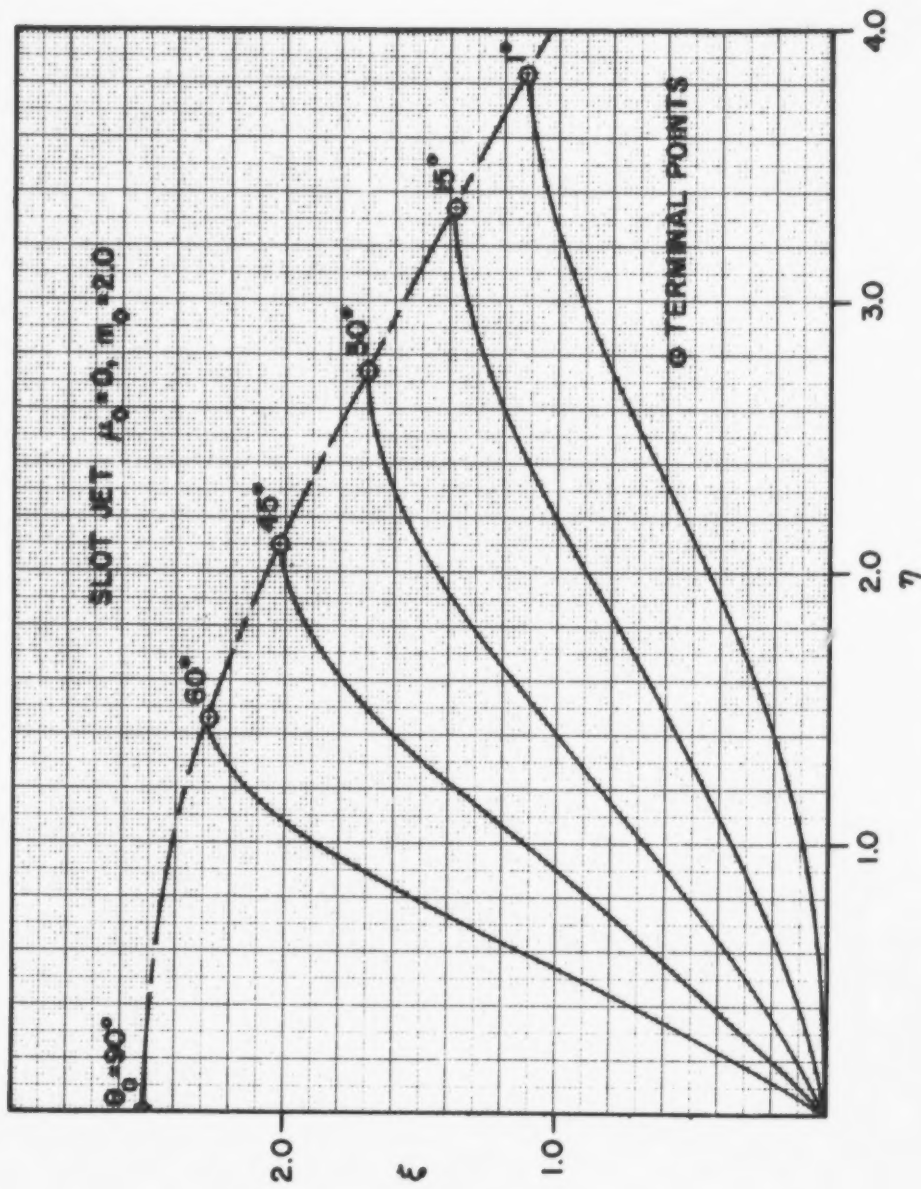


Fig. 21. Paths of Inclined Slot Buoyant Jets with $\mu_0 = 0, m_0 = 0.2$.

The variation of the terminal height of rise ε_t for the range $\mu_o = 0$ to 0.01 is shown in Fig. 22. The curves for $m_o = 2.0$ shows a faster rate of decrease in ε_t as the angle θ_o decreases than the cases for smaller m_o values. For a simple plume where m_o vanishes, the height ($\varepsilon_t = 2.96$) is independent of the initial angle of discharge. Fig. 23 shows the variation of the terminal volume flux parameter μ_t over the same range of initial conditions. A substantial increase in μ_t can be achieved by decreasing the angle θ_o at large m_o values. Except for ε_t values at small m_o values ($m_o \sim 0$ to 0.02), Figs. 22 and 23 demonstrate that the variation of the terminal quantities ε_t and μ_t are independent of the initial volume flux parameter μ_o in the range 0 to 0.01. Since the dilution ratio S_o at the terminal point is the square root of the ratio of the terminal and initial volume flux parameters, an increase in the terminal dilution ratio can be obtained by decreasing μ_o (such as by using a long diffuser).

Numerical solutions for both horizontal and vertical cases were carried out covering a wide range of initial conditions. The variations of ε_t and μ_t for vertical jets ($\theta_o = 90^\circ$) are shown in Figs. 24 and 25. The variations of ε_t , η_t and μ_t values for horizontal jets ($\theta_o = 0^\circ$) are shown in Figs. 26, 27 and 28.

VI-D. Application to Practical Problems

For application to a practical problem the zone of flow establishment is again assumed to follow Albertson et al.'s (5) results as given in Sec. V-D (Fig. 22). Eqs. (104) to (107), (112) and (113) are assumed to be applicable to a slot buoyant jet in a linearly stratified environment as well as a uniform environment.

The dilution ratio S_o with respect to initial concentration at O is

$$S_o = c_o / c = \sqrt{\mu / \mu_o} \quad (128)$$

The dilution ratio S with respect to the initial concentration at O' is

$$S = \sqrt{\left(\frac{2\lambda^2}{1+\lambda^2}\right) \mu / \mu_o} = \sqrt{\frac{2\lambda^2}{1+\lambda^2}} S_o \quad (129)$$

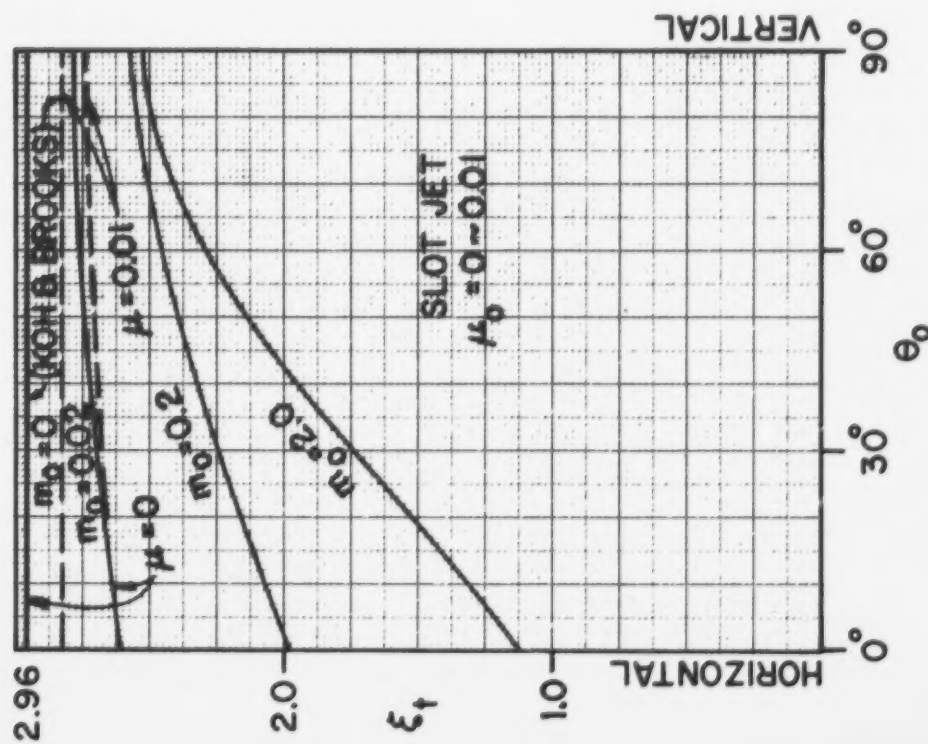


Fig. 22. Terminal Height of Rise ξ_t for Inclined Slot Buoyant Jets with $\mu_0 = 0$ to 0.01.

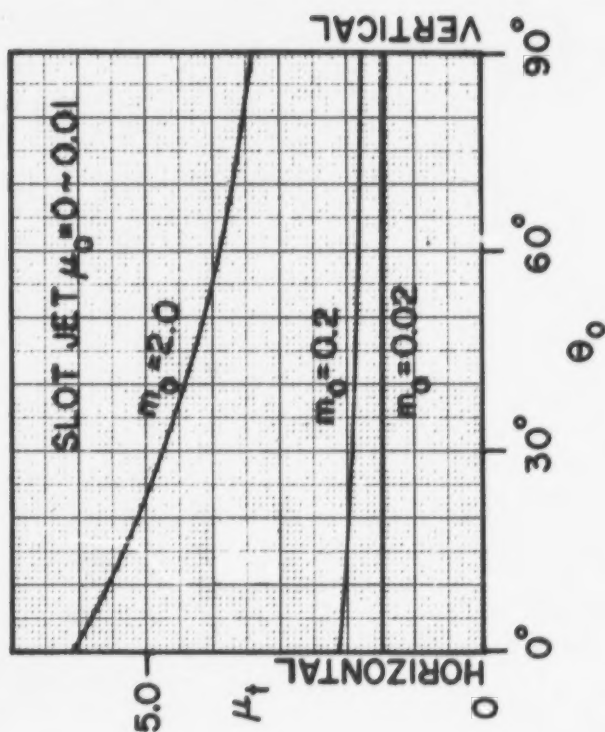


Fig. 23. Terminal Volume Flux Parameter μ_t for Inclined Slot Buoyant Jets with $\mu_0 = 0$ to 0.01.

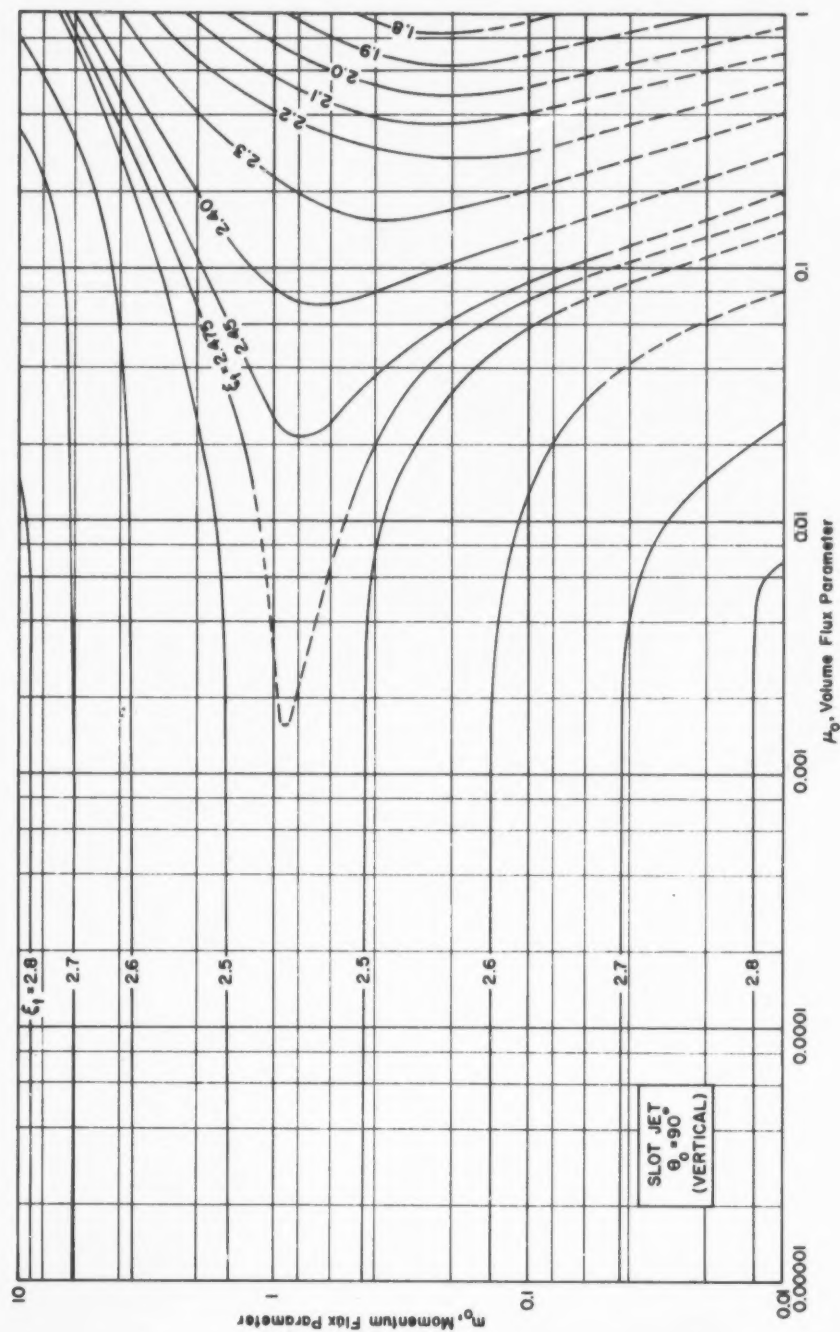


Fig. 24. Terminal Height of Rise ξ_t for Slot Buoyant Jets ($\theta_o = 90^\circ$, vertical) in Linearly Density-Stratified Environments.

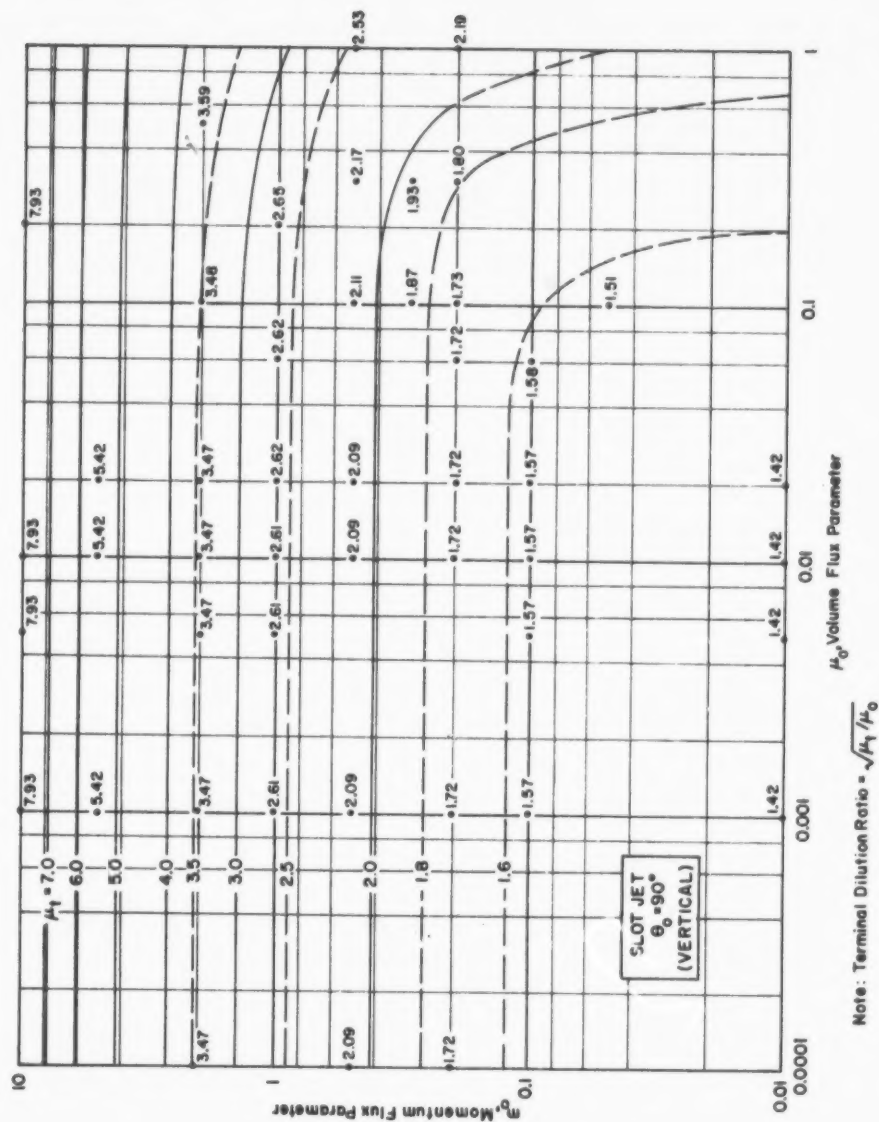


Fig. 25. Terminal Volume Flux Parameter μ_t for Slot Buoyant Jets ($\theta_o = 90^\circ$, vertical) in Linearly Density-Stratified Environments.

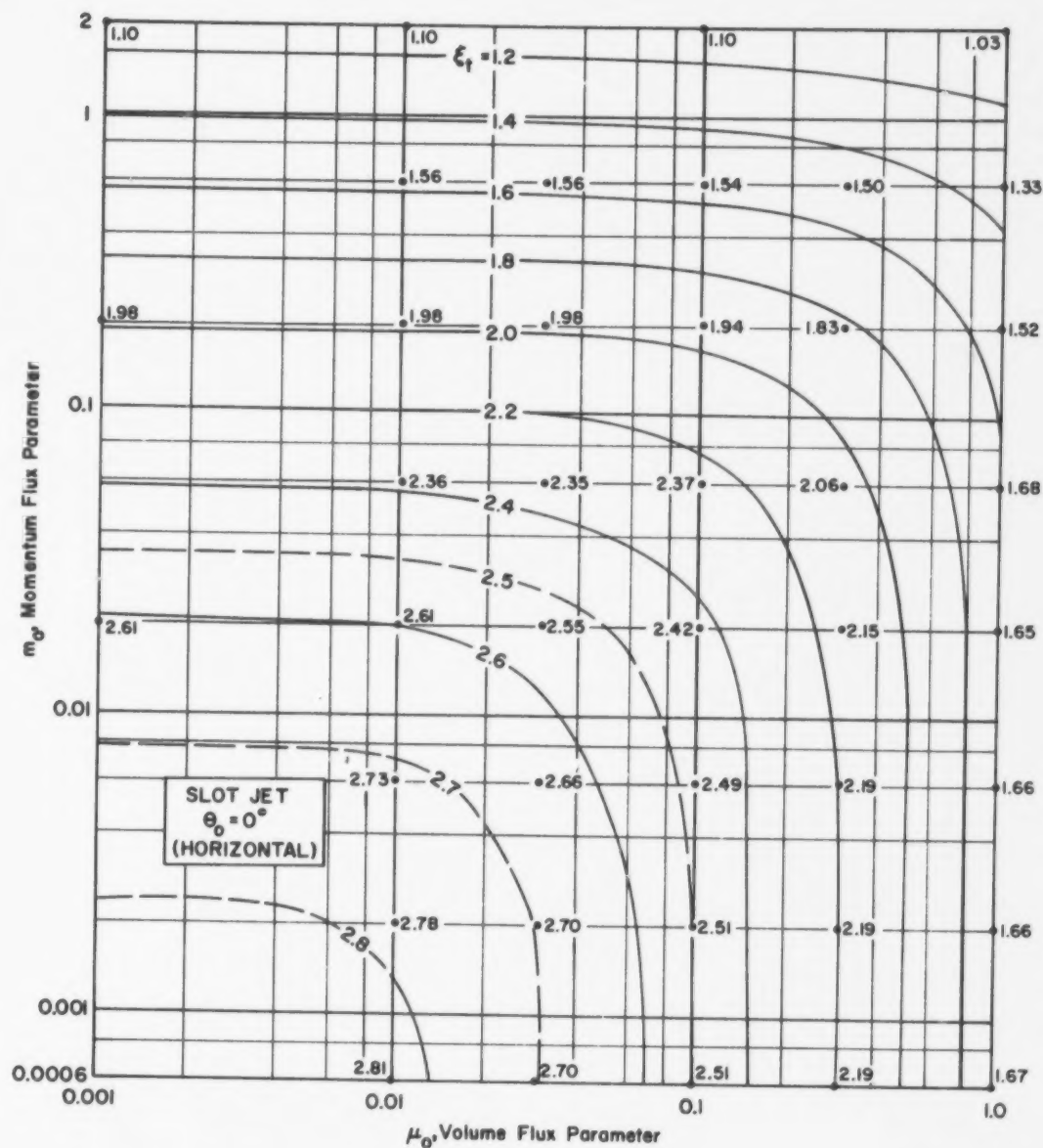


Fig. 26. Terminal Height of Rise ξ_t for Slot Buoyant Jets ($\theta_0 = 0^\circ$, horizontal) in Linearly Density-Stratified Environments.

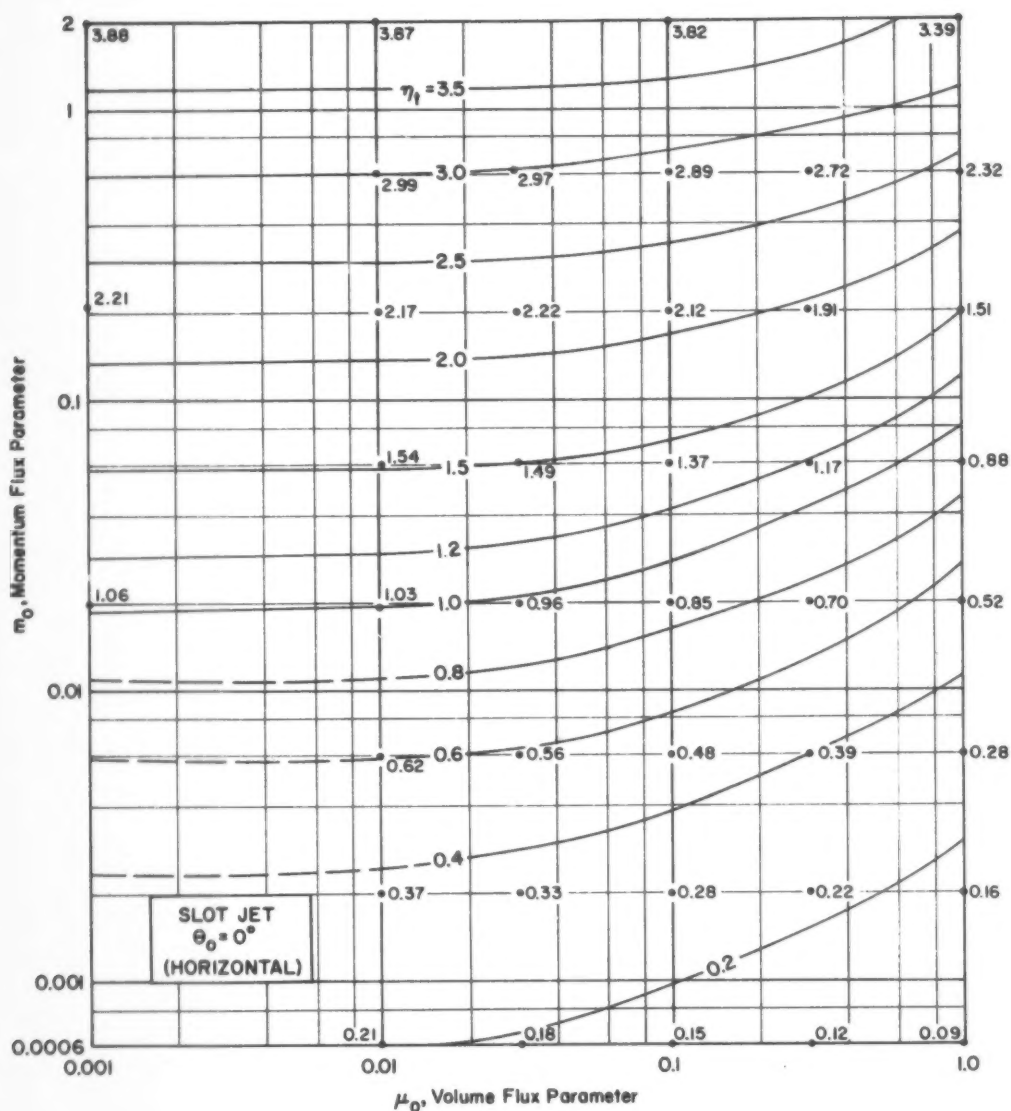


Fig. 27. Terminal Horizontal Coordinate η_t for Slot Buoyant Jets ($\theta_0 = 0^\circ$, horizontal) in Linearly Density-Stratified Environments.

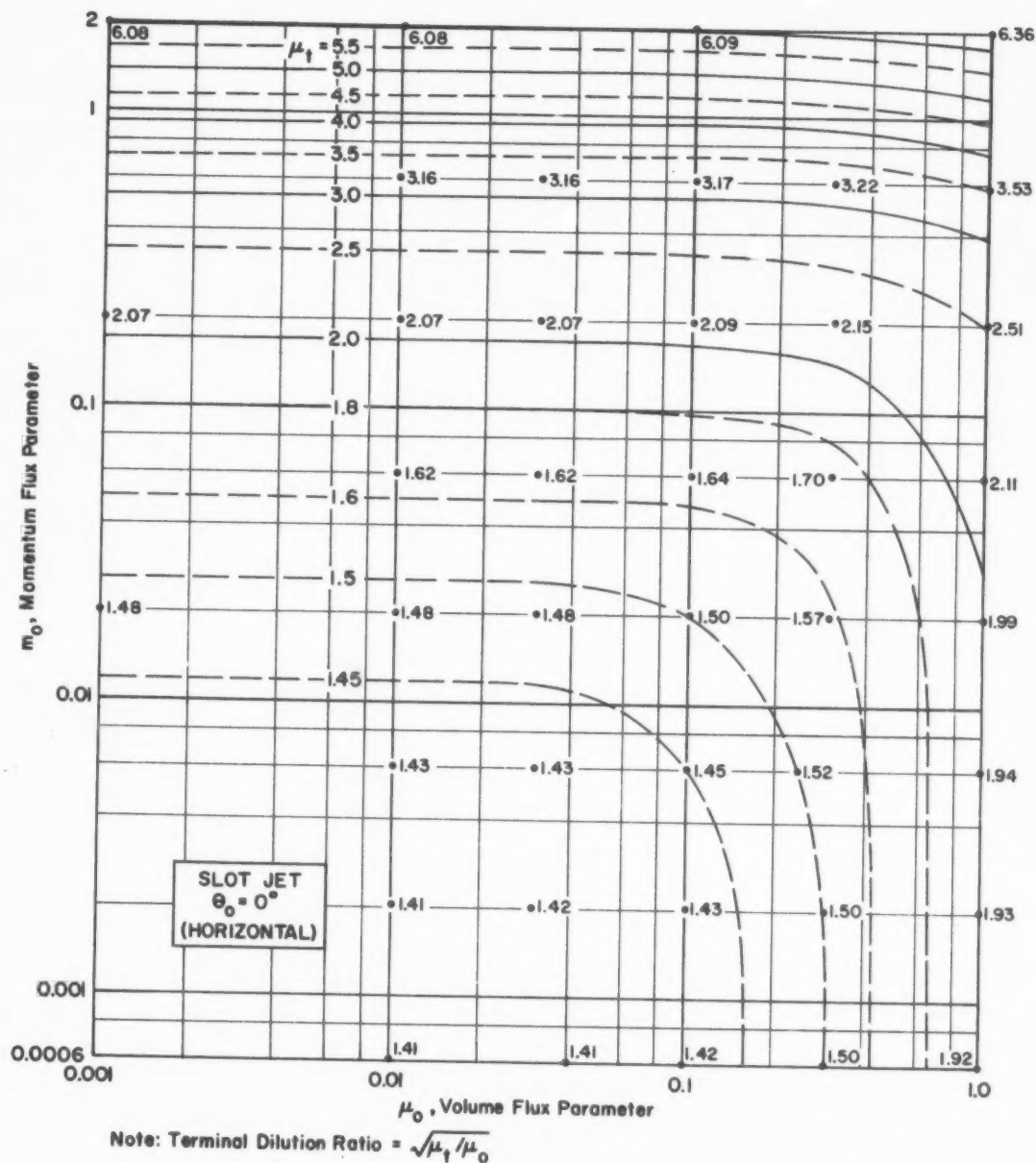


Fig. 28. Terminal Volume Flux Parameter μ_t for Slot Buoyant Jets ($\theta_0 = 0^\circ$, horizontal) in Linearly Density-Stratified Environments.

except for $\lambda < 1$, when we take $S = 1$ when $S_o \sqrt{\frac{2\lambda^2}{1+\lambda^2}} < 1$.

A densimetric jet Froude number F is defined as:

$$F = \frac{U_o}{\sqrt{\frac{\rho_o - \rho_1}{\rho_o} g B}}, \quad (130)$$

and a stratification parameter T is defined as:

$$T = \frac{\rho_o - \rho_1}{B \left(\frac{d\rho_a}{dy} \right)}, \quad (131)$$

where, as before, B is the width of the jet issuing from a slot, and the initial length parameter b_o in the solution is given by

$$b_o = \sqrt{\frac{2}{\pi}} B \quad (105)$$

It can be shown that the problem of an inclined slot buoyant jet in a linearly stratified environment is characterized by the three parameters, F , T , and θ_o . The relationships between the set of parameters F , T and the set m_o, μ_o are:

$$m_o = \left(\frac{\sqrt{2(1+\lambda^2)}}{4\lambda^2} \right) F^2 / T \quad (132)$$

$$\mu_o = \left(\frac{\sqrt{2(1+\lambda^2)}}{4\lambda^{4/3} \alpha^{2/3}} \right) F^{2/3} / T \quad (133)$$

or,
$$F = (\lambda/\alpha)^{1/2} (m_o/\mu_o)^{3/4} \quad (134)$$

$$T = \left(\frac{\sqrt{2(1+\lambda^2)}}{4\lambda\alpha} \right) m_o^{1/2} / \mu_o^{3/2} \quad (135)$$

The set of parameters m_o and μ_o has been adopted in the analysis for mathematical simplicity, and because the solutions found are independent of α and λ values chosen. But in practical application it is more convenient to use F and T as parameters.

A scale factor δ defined by:

$$x/B = \delta \eta \text{ and } y/B = \delta \xi \quad (136)$$

is related to m_o and μ_o for F and T by

$$\delta = \frac{1}{2^{3/2} \alpha} m_o^{1/2} / \mu_o = \left[\frac{\lambda^{1/3}}{2^{3/4} \alpha^{1/3} (1+\lambda^2)^{1/4}} \right] F^{1/3} T^{1/2} \quad (137)$$

The coordinate distances from O' are again given by eqs. (112) and (113). The dimensionless half width of the jet $\tilde{w}$ is given by:

$$\begin{aligned} \tilde{w} &= \left\{ 32\sqrt{2(1+\lambda^2)} G^3 \lambda^2 / F_o^2 \pi \right\}^{1/6} (\sqrt{2} b) \\ &= 8\alpha\mu / \sqrt{\pi h_o} \quad \text{for } h_o \neq 0 \end{aligned} \quad (138)$$

At the terminal point,

$$S_t = \sqrt{\frac{2\lambda^2}{1+\lambda^2}} (\mu_t / \mu) \quad (139)$$

and

$$\tilde{w}_t = 8\alpha\mu_t / \sqrt{\pi h_o} \quad (140)$$

The suggested values for the coefficient of entrainment α and the turbulent spreading ratio λ are respectively 0.16 and 0.89 based upon Rouse et al.'s (23) data for a simple two-dimensional plume. These values have been confirmed by Lee and Emmons (17) in their study of line fires. For application to jets in stratified environments it is a further generalization. Experimental confirmation is so far (1968) lacking.

With these substitutions ($\alpha = 0.16$ and $\lambda = 0.89$), eqs. (129), (132) to (135), (137) to (140) are:

$$S = 0.94 \sqrt{\mu/\mu_o} = 0.94 S_o$$

$$\text{for } \sqrt{\mu/\mu_o} = S_o > 1.06 \quad (141)$$

$$S = S_o = 1 \text{ for } \sqrt{\mu/\mu_o} < 1.06$$

$$m_o = 0.600 F^2/T \quad (142)$$

$$\mu_o = 1.89 F^{2/3}/T \quad (143)$$

$$F = 2.36 m_o^{3/4}/\mu_o^{3/4} \quad (144)$$

$$T = 3.31 m_o^{1/2}/\mu_o^{3/2} \quad (145)$$

$$\delta = 2.21 m_o^{1/2}/\mu_o = 0.908 F^{1/3} T^{1/2} \quad (146)$$

$$\tilde{w} = 0.72 \mu/\sqrt{h_o} \quad (147)$$

At terminal point,

$$S_t = 0.94 \sqrt{\mu_t/\mu_o} \quad (148)$$

$$\tilde{w}_t = 0.72 \mu_t/\sqrt{h_o} \quad (149)$$

CHAPTER VII

SUMMARY AND CONCLUSIONS

For a buoyant jet (either two- or three-dimensional) in a uniform environment, the independent dimensionless variables are the jet densimetric Froude number F , the vertical distance (y/D or y/B) and the angle of jet discharge (θ_0). In a linearly density-stratified environment, the additional variable is a stratification parameter T . In this report, equations and variables were normalized in such a way that the dimensionless solutions were independent of the numerical values of the entrainment coefficient α and spreading ratio λ , but the final results for any problem do depend on α and λ .

Numerical solutions have been presented for four different classes of buoyant jet problems, namely

- (1) an inclined round buoyant jet in a stagnant uniform environment (Chapter III)
- (2) an inclined round buoyant jet in a stagnant environment with linear density stratification (Chapter IV)
- (3) an inclined two-dimensional slot buoyant jet in a stagnant uniform environment (Chapter V), and
- (4) an inclined two-dimensional slot buoyant jet in a stagnant environment with linear density stratification (Chapter VI).

Morton-type integral analyses were carried out based upon similarity and entrainment assumptions (Chapter II). A wide range of initial conditions were covered in this investigation. Results covering gross jet characteristics are immediately applicable to practical problems. For items (1) and (3) above, the graphs give centerline dilutions, normalized trajectories, and widths are given as functions of normalized heights and Froude numbers for horizontal and vertical jets. For items (2) and (4) graphs present the maximum height of rise

as well as the dilution and horizontal displacement at that top point, as functions of the Froude number F and stratification number T for vertical and horizontal jets.

The following values of α and λ are recommended for buoyant jets based primarily on Rouse, Yih, and Humphreys (23):

	<u>Entrainment coefficient</u>	<u>Spreading ratio</u>
Round (circular)	$\alpha = 0.082$	$\lambda = 1.16$
Slot (two-dimensional)	$\alpha = 0.16$	$\lambda = 0.89$

When more adequate experimental data on these coefficients are available, the present solutions are still applicable by simple substitution of the new values into various normalizing factors. It is presumed however that α and λ stay constant over any one trajectory.

Limitations of these solutions, especially the cases with density stratification, were discussed in detail by Fan (10). In summary, a vertical, or a nearly vertically rising buoyant jet is not predicted well by the analysis near its terminal point because of breakdown of many essential assumptions. For example, the similarity assumptions can no longer hold since the falling fluid tends to interfere with the rising jet near its terminal point.

For cases of non-linear density gradient in the ambient fluid, the density profile may be approximated by a straight line in the pertinent portion. If this is not feasible, then problems of this type are not readily treated in dimensionless form. Rather it is easier to make case-by-case computer calculations using dimensional variables. A detailed program in FORTRAN language for doing this is available in a technical memorandum of the Keck Laboratory by Ditmars (16).

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APPENDIX

SUMMARY OF NOTATIONS

a	subscripts denoting quantities of the ambient fluid
A	jet cross section normal to the jet axis
b	local characteristic length or half width of the jet
b _o	half width of the jet at the end of zone of flow establishment
B	initial slot jet width
B _e	equivalent slot width
c _o	initial concentration at the end of the zone of flow establishment, i. e. at O, on jet axis
c	concentration at jet axis
c*	local concentration in jet
D	initial diameter of circular jet
D _o	diameter of a sharp-edged orifice
f	dimensionless buoyancy parameter
F	densimetric jet Froude number
F _o	initial jet buoyancy flux
g	gravitational acceleration
G	$-\frac{g}{\rho_o} \frac{d\rho_a}{dy}$
h	dimensionless horizontal or x-momentum flux parameter
L	port spacing on centers
m	dimensionless momentum flux parameter
m _o	initial dimensionless momentum flux parameter
o	subscript denoting initial values
O	origin of the coordinate systems (x, y), beginning of the zone of established flow
O'	origin of the coordinate systems (x', y'), point of the jet discharge
r	radial distance measured from the jet axis on A
s	parametric distance along jet axis
S _o	center-line dilution ratio = c _o /c
S	center-line dilution ratio with respect to jet discharge
S _t	terminal dilution ratio, i. e. S at the terminal point

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c*	local concentration in jet
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D _o	diameter of a sharp-edged orifice
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F	densimetric jet Froude number
F _o	initial jet buoyancy flux
g	gravitational acceleration
G	$-\frac{g}{\rho_o} \frac{d\rho_a}{dy}$
h	dimensionless horizontal or x-momentum flux parameter
L	port spacing on centers
m	dimensionless momentum flux parameter
m _o	initial dimensionless momentum flux parameter
o	subscript denoting initial values
O	origin of the coordinate systems (x, y), beginning of the zone of established flow
O'	origin of the coordinate systems (x', y'), point of the jet discharge
r	radial distance measured from the jet axis on A
s	parametric distance along jet axis
S _o	center-line dilution ratio = c _o /c
S	center-line dilution ratio with respect to jet discharge
S _t	terminal dilution ratio, i. e. S at the terminal point

t	subscript denoting values at the terminal point
T	stratification parameter
u	jet velocity at the center-line of the jet
u^*	jet velocity at a local point
U_0	jet discharge velocity
v	dimensionless vertical or y -momentum flux parameter
w	nominal jet half width = $\sqrt{2} b$
w_t	terminal jet half width
$\tilde{w}$	dimensionless jet half width
$\tilde{w}_t$	terminal jet half width (dimensionless)
x	coordinate axis in horizontal direction in the same plane as jet axis with origin at O
x'	coordinate axis in horizontal direction in the same plane as jet axis with origin at O'
x_t	x -coordinate of the terminal point
y	coordinate axis in vertical direction, with origin at O
y'	coordinate axis in vertical direction, with origin at O'
y_t	terminal height of rise; y -coordinate of the terminal point
α	coefficient of entrainment
β	buoyancy flux parameter
ξ	dimensionless vertical distance (y)
ξ_t	dimensionless terminal height of rise (y_t)
η	dimensionless horizontal distance (x)
η_t	dimensionless horizontal distance (x_t)
ζ	dimensionless distance along s -axis (s)
ζ_t	dimensionless distance (s_t)
θ	angle of inclination of the jet axis (with respect to x -axis)
θ_0	initial angle of inclination (with respect to x -axis)
λ	spreading ratio between buoyancy and velocity profiles; or λ^2 is the turbulent Schmidt number
μ	dimensionless volume flux parameter
μ_0	dimensionless initial volume flux parameter
μ_t	dimensionless terminal volume flux parameter

ρ	density along the jet axis
ρ_a	density of the ambient fluid
ρ_0	reference density taken as $\rho_a(0)$
ρ_1	initial jet density
ρ^*	local density within a jet
ρ_a^*	local ambient density
ϖ	angular coordinate on a cross section normal to the jet axis